Sharp upper and lower bounds of the attractor dimension for 3D damped Euler–Bardina equations

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2d damped Euler system on the torus

We first consider the 2d damped/driven Euler system with periodic boundary conditions:

$$\begin{cases} \partial_t u + (u, \nabla_x)u + \gamma u + \nabla_x p = g, \\ \operatorname{div} u = 0, \quad u(0) = u_0. \end{cases}$$

The system is dissipative in $H^1(\mathbb{T}^2)$, $\mathbb{T}^2 = [0, L]^2$ and it is easy to construct a solution of class $L^\infty(0, T, H^1)$. However, the solution in this class is not known to be unique.

$$\int_{\mathbb{T}^2} (u, \nabla_x)u \cdot \Delta_x u \, dx = 0.$$

Navier–Stokes regularization

We add a vanishing viscosity term:

$$\begin{cases} \partial_t u + (u, \nabla_x)u + \nabla_x p + \gamma u = \nu \Delta_x u + g, \\ \operatorname{div} u = 0, \quad u(0) = u_0,\ x \in \mathbb{T}^2. \end{cases}$$

Then for $\nu > 0$ the solution is clearly unique and the semigroup of solution operators is defined $S(t)u_0 = u(t)$. The semigroup $S(t) : H \to H$

$$H := L^2 \cap \{\mathrm{div}\, u = 0\}$$

has a global attractor in H.

Global attractor

Definition

Let $S(t)$, $t \geq 0$, be a semigroup in a Banach space H. The set $\mathscr{A} \subset H$ is a global attractor of the semigroup $S(t)$ if
1) The set $\mathscr{A}$ is compact in H.
2) It is strictly invariant: $S(t)\mathscr{A} = \mathscr{A}$.
3) It attracts the images of bounded sets in H as $t \to \infty$, i.e., for every bounded set $B \subset H$ and every neighborhood $\mathcal{O}(\mathscr{A})$ of the set $\mathscr{A}$ in H there exists $T = T(B, \mathcal{O})$ such that for all $t \geq T$

$$S(t)B \subset \mathcal{O}(\mathscr{A}).$$

Theorem (Ilyin, Miranville, Titi, 2004; Ilyin, Laptev, 2016)

The system

$$\begin{cases} \partial_t u + (u, \nabla_x)u + \nabla_x p + \gamma u = \nu \Delta_x u + g, \\ \operatorname{div} u = 0, \quad u(0) = u_0, \ x \in \mathbb{T}^2 \end{cases}$$

has a global attractor in H with finite fractal dimension satisfying the following two sided order sharp (as $\nu \to 0^+$) estimate

$$1.5 \cdot 10^{-6} \frac{\| \operatorname{curl} g\|^2_{L^2}}{\nu\gamma^3} \le \dim_F \mathscr{A}_\nu \le \frac{3\pi}{256} \frac{\| \operatorname{curl} g\|^2_{L^2}}{\nu\gamma^3},$$

Damped/driven regularized Euler equations

We shall be dealing with a different approximation of the damped Euler system, namely, the so-called inviscid damped Euler–Bardina model

$$\begin{cases} \partial_t u + (\bar{u}, \nabla_x)\bar{u} + \gamma u + \nabla_x p = g, \\ \operatorname{div} u = 0, \quad u(0) = u_0, \quad \bar{u} = (1 - \alpha\Delta_x)^{-1} u. \end{cases}$$

The system is studied for $d = 2, 3$

1) on the torus $\Omega = \mathbb{T}^d = [0, L]^d$. In this case the standard zero mean condition is imposed on u, $\bar{u}$ and g;

2) in the whole space $\Omega = \mathbb{R}^d$;

3) in a bounded domain $\Omega \subset \mathbb{R}^d$ with Dirichlet boundary conditions for $\bar{u}$. Then $\bar{u}$ is recovered from u solving the Stokes problem

$$\begin{cases} (1 - \alpha\Delta_x)\bar{u} + \nabla_x q = u, \\ \operatorname{div} \bar{u} = 0, \quad \bar{u}|_{\partial\Omega} = 0. \end{cases}$$

Here $\alpha = \alpha' L^2$ and $\alpha' > 0$ is a small dimensionless parameter, so that $\bar{u}$ is a smoothed (filtered) vector field.

The phase space with respect to $\bar{u}$ is the Sobolev space $\mathbf{H}^1$ with divergence free condition

$$\bar{u} \in \mathbf{H}^1 := \begin{cases} \dot{\mathbf{H}}^1(\mathbb{T}^d), & x \in \mathbb{T}^d, \ \int_{\mathbb{T}^d} \bar{u}(x)dx = 0, \\ \mathbf{H}^1(\mathbb{R}^d), & x \in \mathbb{R}^d, \\ \mathbf{H}^1_0(\Omega), & x \in \Omega, \end{cases} \qquad \operatorname{div} \bar{u} = 0,$$

and in terms of u, respectively,
$u \in \mathbf{H}^{-1} := (1 - \Delta_x)\mathbf{H}^1 = \mathbf{H}^{-1} \cap \{\operatorname{div} u = 0\}$.
We write the equation as an evolution equation in $\mathbf{H}^1$:

$$\begin{aligned} &\partial_t \bar{u} + B(\bar{u}, \bar{u}) + \gamma \bar{u} = \bar{g}, \\ &\operatorname{div} \bar{u} = 0, \quad \bar{u}(0) = \bar{u}_0, \quad u = (1 - \alpha\Delta_x)\bar{u}, \end{aligned}$$

where

$$B(\bar{u}, \bar{v}) = (1 - \alpha\Pi\Delta_x)^{-1}((\bar{u}, \nabla_x)\bar{v}),$$

Π — is the Helmholtz–Leray projection, $\Pi\Delta_x$ — is the Stokes operator.

Evolution equation in $\mathbf{H}^1$

The bilinear operator B is smoothing in $\mathbf{H}^1$:

$$B:\mathbf{H}^1\times\mathbf{H}^1\to\mathbf{H}^{2-\varepsilon},\ \varepsilon>0,\ d=2,\qquad B:\mathbf{H}^1\times\mathbf{H}^1\to\mathbf{H}^{3/2},\ d=3.$$

The equation is written as an evolution equation in $\mathbf{H}^1$ with *bounded* coefficients. Hence, the local in time existence of a unique solution is straightforward consequence of the Banach contraction principle, and the global existence follows from the a priori estimate

$$\|\bar u(t)\|^2_\alpha\le\|\bar u(0)\|^2_\alpha e^{-\gamma t}+\frac{1}{\gamma^2}\|g\|^2_{L^2},$$

where

$$\|\bar u\|^2_\alpha:=\|\bar u\|^2_{L^2}+\alpha\|\nabla_x\bar u\|^2_{L^2}.$$

Thus, the solution semigroup $S(t):\mathbf{H}^1\to\mathbf{H}^1$, $S(t)\bar u_0=\bar u(t)$ is well defined, where $\bar u(t)$ is the solution at time t.

Main result 1: upper bounds

Theorem

Let $d = 2$. In each case of BC the system possesses a global attractor $\mathscr{A} \Subset \mathbf{H}^1$ with finite fractal dimension satisfying

$$\dim_F \mathscr{A} \le \frac{1}{8\pi} \cdot \begin{cases} \dfrac{1}{\alpha\gamma^4} \min\left(\|\operatorname{rot} g\|^2_{L^2}, \dfrac{\|g\|^2_{L^2}}{2\alpha} \right), & x \in \mathbb{T}^2,\ x \in \mathbb{R}^2, \\ \dfrac{\|g\|^2_{L^2}}{2\alpha^2\gamma^4}, & x \in \Omega \subset \mathbb{R}^2. \end{cases}$$

In the 3D case $d = 3$ the estimate in all tree cases looks formally the same

$$\dim_F \mathscr{A} \le \frac{1}{12\pi} \frac{\|g\|^2_{L^2}}{\alpha^{5/2}\gamma^4}.$$

Main result 2: Kolmogorov flows and lower bounds

The lower bounds are based on the on the instability analysis of the generalized Kolmogorov flows. Let

$$g_s(x_2) = (\gamma\lambda(s)\sin sx_2, 0)^T, \qquad g_s(x_3) = (\gamma\lambda(s)\sin sx_3, 0, 0)^T$$

be the right-hand sides in our system on $\mathbb{T}^2 = [0, 2\pi]^2$ and $\mathbb{T}^3 = [0, 2\pi]^3$, respectively. Here $s \in \mathbb{N}$, $s \gg 1$, and λ is the amplitude. The corresponding staionary solutions are

$$u_s(x_2) = (\lambda(s)\sin sx_2, 0)^T, \qquad u_s(x_3) = (\lambda(s)\sin sx_3, 0, 0)^T.$$

Theorem

For $\lambda \geq \lambda(s)$, where

$$\lambda(s) = c_1 \gamma \frac{(1+\alpha s^2)^2}{s},$$

and c_i are absolute (effectively computable) constants the stationary solutions are unstable and

$$\dim \mathcal{M}^{\mathrm{un}}(u_s) \geq c_2 s^2,\ d=2; \qquad \dim \mathcal{M}^{\mathrm{un}}(u_s) \geq c_3 s^3,\ d=3. \quad (0.1)$$

Corollary

Under the above assumptions

$$\dim_F \mathscr{A} \geq c_6 \begin{cases} \max\left(\dfrac{\|\operatorname{rot} g_s\|^2_{L^2}}{\alpha\gamma^4}, \dfrac{\|g_s\|^2_{L^2}}{\alpha^2\gamma^4}\right), & x \in \mathbb{T}^2, \\ \dfrac{\|g_s\|^2_{L^2}}{\alpha^{5/2}\gamma^4}, & x \in \mathbb{T}^3. \end{cases}$$

Since

$$\mathcal{M}^{\mathrm{un}} \subset \mathscr{A} \quad \Rightarrow \quad \dim \mathscr{A} \geq \dim \mathcal{M}^{\mathrm{un}}$$

it remains to express the number of unstable eigenmodes in terms of the physical parameters. The system is studied in the limit as $\alpha \to 0^+$ We set

$$s = \frac{1}{\sqrt{\alpha}}\,.$$

Then λ and $\|g_s\|^2_{L^2}$ become

$$\lambda = c_4\gamma\sqrt{\alpha}, \qquad \|g_s\|^2_{L^2} = c_5\gamma^4\alpha,$$

and we obtain as a result

$$\dim_F \mathscr{A} \geq c_3 s^3 = c_3\frac{1}{\alpha^{3/2}} = c_3\frac{\alpha\|g_s\|^2_{L^2}}{\alpha^{5/2}\|g_s\|^2_{L^2}} = c_3\frac{\alpha\|g_s\|^2_{L^2}}{\alpha^{5/2}c_5\alpha\gamma^4},$$

which proves the lower bound for $\mathbb{T}^3$.
For $\mathbb{T}^2$ we see that $\lambda \sim \gamma\sqrt{\alpha}$ and $\|\operatorname{rot} g_s\|^2_{L^2} \sim \gamma^4$ and as before $\|g_s\|^2_{L^2} \sim \gamma^4\alpha$. Arguing similarly we obtain the lower bound for $\mathbb{T}^2$.

A priori estimates

Proposition

Let u be a sufficiently regular solution of our equation. Then the following dissipative energy estimate holds:

$$\|\bar{u}(t)\|_\alpha^2 \le \|\bar{u}(0)\|_\alpha^2 e^{-\gamma t} + \frac{1}{\gamma^2}\|g\|_{L^2}^2,$$

where

$$\|\bar{u}\|_\alpha^2 := \|\bar{u}\|_{L^2}^2 + \alpha\|\nabla\bar{u}\|_{L^2}^2.$$

Taking the scalar product with $\bar{u}$, integrating over Ω we obtain

$$\frac{d}{dt}\left(\|\bar{u}\|_{L^2}^2 + \alpha\|\nabla_x\bar{u}\|_{L^2}^2\right) + 2\gamma\left(\|\bar{u}\|_{L^2}^2 + \alpha\|\nabla_x\bar{u}\|_{L^2}^2\right) = 2(g,\bar{u}) \le$$

$$\le 2\|g\|_{L^2}\|\bar{u}\|_{L^2} \le \gamma\|\bar{u}\|_{L^2}^2 + \frac{1}{\gamma}\|g\|_{L^2}^2.$$

Applying the Gronwall inequality we complete the proof.

Corollary

Let u be a sufficiently smooth solution. Then the following estimate holds ($d = 2, 3$ any type of the domain):

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \|\nabla u(s)\|_{L^2}\, ds \le \frac{1}{\gamma\sqrt{2\alpha}}\|g\|_{L^2}.$$

Indeed, integrating the differential inequality over t, taking the limit $t \to \infty$ and using the fact that $\|u(t)\|_\alpha^2$ remains bounded, we arrive at we obtain

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \|\nabla u(s)\|_{L^2}^2\, ds \le \frac{1}{2\alpha\gamma^2}\|g\|_{L^2}^2.$$

Using after that the Hölder inequality

$$\frac{1}{t}\int_0^t \|\nabla u(s)\|_{L^2}\, ds \le \left(\frac{1}{t}\int_0^t \|\nabla u(s)\|_{L^2}^2\, dx\right)^{1/2},$$

we complete the proof.

Use of vorticity equation for $\mathbb{T}^2$ and $\mathbb{R}^2$

Corollary

Let $\Omega = \mathbb{T}^2$ or $\mathbb{R}^2$ and let u be a sufficiently smooth solution of our problem. Then the following estimate holds:

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^1 \|\nabla \bar{u}(s)\|_{L^2}\, ds \le \frac{1}{\gamma} \min\left\{ \|\operatorname{curl} g\|_{L^2}, \frac{\|g\|_{L^2}}{\sqrt{2\alpha}} \right\}.$$

Upper bound

Theorem

Suppose that Ω *is either the 3D torus* $\mathbb{T}^3$*, or a bounded domain* $\Omega \subset \mathbb{R}^3$ *(endowed with Dirichlet BC), or the whole space* $\Omega = \mathbb{R}^3$*. Let* $g \in [L^2(\Omega)]^d$ *(in the case of* $\mathbb{T}^3$ *we also assume that* g *has zero mean). Then the global attractor* $\mathscr{A}$ *corresponding to the regularized damped Euler system has finite fractal dimension satisfying the following estimate:*

$$\dim_F \mathscr{A} \le \frac{1}{12\pi} \frac{\|g\|_{L^2}^2}{\alpha^{5/2}\gamma^4} .$$

The linearized operator:

$$\partial_t\bar\theta = -\gamma\bar\theta - B(\bar u(t),\bar\theta) - B(\bar\theta,\bar u(t)) =: L_{u(t)}\bar\theta,$$
$$\operatorname{div}\bar\theta = 0,\ \bar\theta\big|_{t=0} = \bar\theta_0 \in \mathbf{H}^1(\Omega),$$

where $B(\bar u,\bar v) := \Pi A_\alpha\left((\bar u,\nabla_x)\bar v\right)$. In order to utilize the well-known cancelation property

$$(\bar u,\nabla_x)\bar\theta,\bar\theta) \equiv 0$$

for the inertial term in the Navier-Stokes equations, it is natural to endow the space $\mathbf{H}^1$ with the scalar product

$$(\bar\theta,\bar\xi)_\alpha = (\bar\theta,\bar\xi) + \alpha(\nabla_x\bar\theta,\nabla_x\bar\xi) = ((1-\alpha A)\bar\theta,\bar\xi).$$

Then we get the cancelation

$$(B(\bar u,\bar\theta),\bar\theta)_\alpha = \Big((1-\alpha\Delta_x)^{-1}(\bar u,\nabla_x)\bar\theta,(1-\alpha\Delta_x)\bar\theta\Big) = ((\bar u,\nabla_x)\bar\theta,\bar\theta) \equiv 0$$

We estimate the global Lyapunov exponents which control the dimension:

$$q(n) := \limsup_{t\to\infty} \sup_{u(t)\in\mathscr{A}} \sup_{\{\bar\theta_j\}_{j=1}^n} \frac{1}{t}\int_0^t \sum_{j=1}^n (L_{u(\tau)}\bar\theta_j, \bar\theta_j)_\alpha d\tau,$$

where the first (inner) supremum is taken over all orthonormal families $\{\bar\theta_j\}_{j=1}^n$ with respect to the scalar product $(\cdot,\cdot)_\alpha$ in $\mathbf{H}^1$:

$$(\bar\theta_i, \bar\theta_j)_\alpha = \delta_{ij}, \quad \operatorname{div}\theta_j = 0,$$

and the second (middle) supremum is over all trajectories $u(t)$ on the attractor $\mathscr{A}$. Then, using the cancellation mentioned above together with the pointwise estimate

$$\sum_{j=1}^n (L_{u(t)}\bar\theta_j, \bar\theta_j)_\alpha = -\sum_{j=1}^n \gamma\|\bar\theta_j\|_\alpha^2 - \sum_{j=1}^n ((\bar\theta_j, \nabla_x)\bar u, \bar\theta_j) \le$$

$$\le -\gamma n + \sqrt{\frac{2}{3}}\int_\Omega \rho(x)|\nabla_x \bar u(t,x)|dx \le -\gamma n + \sqrt{\frac{2}{3}}\|\nabla_x \bar u(t)\|_{L^2}\|\rho\|_{L^2},$$

where

$$\rho(x) = \sum_{j=1}^{n} |\bar\theta_j(x)|^2.$$

We now use estimate

$$\|\rho\|_{L^2} \le \frac{1}{2\sqrt{\pi}} \frac{n^{1/2}}{\alpha^{3/4}}$$

and obtain

$$\sum_{j=1}^{n} (L_{u(t)}\bar\theta_j, \bar\theta_j)_\alpha \le -\gamma n + \frac{1}{\sqrt{6}\pi} \frac{n^{1/2}}{\alpha^{3/4}} \|\nabla_x \bar u(t)\|_{L^2}.$$

Finally, using the estimate on the attractor , we arrive at

$$q(n) \le -\gamma n + \frac{1}{2\sqrt{3\pi}} \frac{n^{1/2}}{\alpha^{5/4}} \frac{\|g\|_{L^2}}{\gamma}.$$

It only remains to recall that, according to the general theory, any number n^* for which $q(n^*) \le 0$ an upper bound both for the Hausdorff and the fractal dimension of the global attractor $\mathscr{A}$. This gives the desired estimate

$$q(n) \le -\gamma n + \frac{1}{2\sqrt{3\pi}} \frac{n^{1/2}}{\alpha^{5/4}} \frac{\|g\|_{L^2}}{\gamma}.$$

Therefore

$$\dim_F \mathscr{A} \le \frac{1}{12\pi} \frac{\|g\|_{L^2}^2}{\alpha^{5/2}\gamma^4}$$

Spectral inequalities

Theorem

Let $\Omega \subseteq \mathbb{R}^d$ *be an arbitrary domain. Let a family of vector functions* $\{\bar\theta_i\}_{i=1}^n \in \mathbf{H}^1(\Omega)$ *with* $\operatorname{div}\bar\theta_i = 0$ *be orthonormal with respect to the scalar product*

$$m^2(\bar\theta_i,\bar\theta_j)_{L^2} + (\nabla\bar\theta_i,\nabla\bar\theta_j)_{L^2} = m^2(\bar\theta_i,\bar\theta_j)_{L^2} + (\operatorname{curl}\bar\theta_i,\operatorname{curl}\bar\theta_j)_{L^2} = \delta_{ij},$$

Then the function $\rho(x) := \sum_{j=1}^n |\bar\theta_j(x)|^2$ *satisfies*

$$\|\rho\|_{L^2} \le \frac{1}{2\sqrt{\pi}}\frac{n^{1/2}}{m}, \qquad d=2,$$

$$\|\rho\|_{L^2} \le \frac{1}{2\sqrt{\pi}}\frac{n^{1/2}}{m^{1/2}}, \qquad d=3.$$

We first let $\Omega = \mathbb{R}^d$ and introduce the operators

$$\mathbb{H} = V^{1/2}(m^2 - \Delta_x)^{-1/2}\Pi, \quad \mathbb{H}^* = \Pi(m^2 - \Delta_x)^{-1/2}V^{1/2}$$

acting in $[L^2(\mathbb{R}^d)]^d$, where $V \in L^1(\mathbb{R}^d)$ is a non-negative scalar function which will be specified below and Π is the Helmholtz–Leray projection. We define a compact self-adjoint operator $\mathbf{K}$

$$\mathbf{K} = \mathbb{H}^*\mathbb{H} : [L^2(\mathbb{R}^d)]^d \to [L^2(\mathbb{R}^d)]^d.$$

Then

$$\begin{aligned}\operatorname{Tr}\mathbf{K}^2 &= \operatorname{Tr}\left(\Pi(m^2 - \Delta_x)^{-1/2}V(m^2 - \Delta_x)^{-1/2}\Pi\right)^2 \le \\ &\le \operatorname{Tr}\left(\Pi(m^2 - \Delta_x)^{-1}V^2(m^2 - \Delta_x)^{-1}\Pi\right) = \\ &= \operatorname{Tr}\left(V^2(m^2 - \Delta_x)^{-2}\Pi\right),\end{aligned}$$

where we used the Araki–Lieb–Thirring inequality for traces

$$\operatorname{Tr}(BA^2B)^p \le \operatorname{Tr}(B^pA^{2p}B^p), \quad p \ge 1,$$

and the cyclicity property of the trace together with the facts that Π commutes with the Laplacian and that Π is a projection: $\Pi^2 = \Pi$.

We want to show that

$$\operatorname{Tr}\mathbf{K}^2 \le \begin{cases} \frac{1}{4\pi}\frac{1}{m^2}\|V\|^2_{L^2}, & d=2;\\ \frac{1}{4\pi}\frac{1}{m}\|V\|^2_{L^2}, & d=3. \end{cases}$$

The fundamental solution of $(m^2-\Delta_x)^2\Pi$ in $\mathbb{R}^d$ is a $d\times d$ matrix

$$\mathbf{F}^d_{ij}(x) = G_d(x)\delta_{ij} - \partial_{x_i}\partial_{x_j}\Delta^{-1}G_d(x)$$

with $\mathbb{R}^d$-trace at $x\in\mathbb{R}^d$

$$\operatorname{Tr}_{\mathbb{R}^d}\mathbf{F}^d(x) = dG_d(x) - \sum_{i=1}^d \partial^2_{x_ix_i}\Delta_x^{-1}G_d(x) = (d-1)G_d(x),$$

where $G_d(x)$ is a fundamental solution of the scalar operator $(m^2-\Delta_x)^2$ in the whole space $\mathbb{R}^d$:

$$G_d(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{e^{i\xi x}d\xi}{(m^2+|\xi|^2)^2} = \begin{cases} \frac{1}{8\pi}\frac{1}{m}e^{-|x|m}, & d=3;\\ \frac{1}{4\pi}\frac{1}{m^2}|x|mK_1(|x|m), & d=2. \end{cases}$$

Stein, Watson

$$G_2(x)=\frac{1}{2\pi}\mathscr{F}^{-1}\left((m^2+|\xi|^2)^2\right)=\frac{1}{2\pi}\int_0^\infty\frac{J_0(|x|r)rdr}{(m^2+r^2)^2}=\frac{1}{4\pi}\frac{1}{m^2}|x|mK_1(|x|m),$$

where K_1 is the modified Bessel function of the second kind. Thus, the operator $V^2(m^2-\Delta_x)^2\Pi$ has the matrix-valued integral kernel

$$V(y)^2\mathbf{F}^d(x-y)$$

and therefore

$$\mathrm{Tr}(V^2(m^2-\Delta_x)^2\Pi)=$$

$$=\int_{\mathbb{R}^d}\mathrm{Tr}_{\mathbb{R}^d}\left(V(y)^2\,\mathbf{F}^d(0)\right)dy=(d-1)\|V\|^2_{L^2}G_d(0)$$

which the first inequality , and also the second one, since $(tK_1(t))|_{t=0}=1$.

We can now complete the proof as in in the original paper by E.Lieb. Setting

$$\psi_i := (m^2 - \Delta_x)^{1/2}\bar{\theta}_i,$$

we see that $\{\psi_j\}_{j=1}^n$ is an orthonormal family in L^2. We observe that

$$\int_{\mathbb{R}^d} \rho(x)V(x)dx = \sum_{i=1}^n \|\mathbb{H}\psi_i\|_{L^2}^2,$$

and in view of orthonormality of the ψ_j's in L^2 we obtain

$$\sum_{i=1}^n \|\mathbb{H}\psi_i\|_{L^2}^2 = \sum_{i=1}^n (\mathbf{K}\psi_i, \psi_i) \le \sum_{i=1}^n \|\mathbf{K}\psi_i\|_{L^2} \le n^{1/2}\left(\sum_{i=1}^n \|\mathbf{K}\psi_i\|_{L^2}^2\right)^{1/2} =$$

$$= n^{1/2}\left(\sum_{i=1}^n (\mathbf{K}^2\psi_i, \psi_i)\right)^{1/2} \le n^{1/2}\left(\operatorname{Tr}\mathbf{K}^2\right)^{1/2}.$$

This gives

$$\int_{\mathbb{R}^d} \rho(x)V(x)dx \le n^{1/2}\left(\operatorname{Tr}\mathbf{K}^2\right)^{1/2}.$$

Setting $V(x) := \rho(x)$ and using

$$\operatorname{Tr}\mathbf{K}^2 \le \begin{cases} \dfrac{1}{4\pi}\dfrac{1}{m^2}\|V\|^2_{L^2}, & d=2; \\ \dfrac{1}{4\pi}\dfrac{1}{m}\|V\|^2_{L^2}, & d=3. \end{cases}$$

we complete the proof of for the case of $\Omega = \mathbb{R}^d$, $d = 2, 3$.
For a proper domain we use extension be zero which works nicely here. The theorem is proved.

Corollary

Let the assumptions of the Theorem hold and let $\{\bar\theta_j\}_{j=1}^n$, $\operatorname{div}\bar\theta_j = 0$ *be an orthonormal system with respect to*

$$(\bar\theta_i, \bar\theta_j)_{L^2} + \alpha(\nabla\bar\theta_i, \nabla\bar\theta_j)_{L^2} = \delta_{ij}.$$

Then $\rho(x) = \sum_{j=1}^n |\bar\theta_j(x)|^2$ *satisfies*

$$\|\rho\|_{L^2} \le \frac{1}{2\sqrt{\pi}}\frac{n^{1/2}}{\alpha^{1/2}}, \qquad d = 2,$$

$$\|\rho\|_{L^2} \le \frac{1}{2\sqrt{\pi}}\frac{n^{1/2}}{\alpha^{3/4}}, \qquad d = 3.$$

Indeed, this statement follows from that with m^2 by the proper scaling.

Spectral inequalities on $\mathbb{T}^2$ and $\mathbb{T}^3$

Now $G_d(x) = G_{d,m}(x)$ is the fundamental solution of the scalar operator $(m^2 - \Delta_x)^{-2}$ on the torus $\mathbb{T}^d$ (with zero mean condition), so the integral should be replaced by the corresponding sum over the lattice $\mathbb{Z}_0^d = \mathbb{Z}^d \setminus \{0\}$:

$$G_d(x) = \frac{1}{(2\pi)^d} \sum_{k \in \mathbb{Z}_0^d} \frac{e^{ik.x}}{(m^2 + |k|^2)^2}$$

and we have to show that

$$G_{d,m}(0) < \begin{cases} \dfrac{1}{8\pi}\dfrac{1}{m}, & d = 3; \\ \dfrac{1}{4\pi}\dfrac{1}{m^2}, & d = 2. \end{cases}$$

In other words, we have to show that

Estimates for lattice sums

$$m\sum_{k\in\mathbb{Z}_0^3}\frac{1}{(|k|^2+m^2)^2}<\pi^2\qquad d=3$$

$$m^2\sum_{k\in\mathbb{Z}_0^2}\frac{1}{(|k|^2+m^2)^2}<\pi\qquad d=2.$$

Lemma

For $m \geq 0$

$$F(m) := m^2 \sum_{k \in \mathbb{Z}_0^2} \frac{1}{(|k|^2 + m^2)^2} < \pi.$$

We assume that $m \geq 1$. We show below that the inequality holds for $m \geq 1$, which proves the Lemma, since $F'(m) > 0$ on $m \in (0, 1]$ and F is increasing on $m \in [0, 1]$.

We use the Poisson summation formula

$$\sum_{k \in \mathbb{Z}^d} f(k/m) = (2\pi)^{d/2} m^d \sum_{k \in \mathbb{Z}^d} \widehat{f}(2\pi k m),$$

where $\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-i\xi x} dx$. For the function $f(x) = 1/(1 + |x|^2)^2$, $x \in \mathbb{R}^2$, with $\int_{\mathbb{R}^2} f(x) dx = \pi$, this gives

$$F(m) = \frac{1}{m^2} \sum_{k \in \mathbb{Z}^2} f(k/m) - \frac{1}{m^2} f(0) = \pi - \frac{1}{m^2} + 2\pi \sum_{k \in \mathbb{Z}_0^2} \widehat{f}(2\pi m k).$$

Since f is radial we have

$$\widehat{f}(\xi) = \int_0^\infty \frac{J_0(|\xi|r)rdr}{(1+r^2)^2} = \frac{|\xi|}{2} K_1(|\xi|),$$

where K_1 is the modified Bessel function of the second kind. Therefore we have to show that

$$\sum_{k \in \mathbb{Z}_0^2} G(2\pi m|k|) < \frac{1}{m^2}, \quad G(x) = \pi x K_1(x).$$

Next, we use the estimate

$$K_1(x) < \left(1 + \frac{1}{2x}\right) \sqrt{\frac{\pi}{2x}} e^{-x}, \quad x > 0,$$

which gives

$$G(2\pi m|k|) < \pi \left(\pi \sqrt{m|k|} + \frac{1}{4\sqrt{m|k|}}\right) e^{-2\pi m|k|}.$$

For the first term we use that

$$\sqrt{x}e^{-ax} \le \frac{1}{\sqrt{2ea}}$$

with $a = \frac{1}{2}\pi m$ and $x = |k|$ (and keep three quarters of the negative exponent), while for the second term we just replace $1/\sqrt{m|k|}$ by 1, since $m \ge 1$ and $k \ge 1$. This gives

$$G(2\pi m|k|) < \pi\left(\sqrt{\frac{\pi}{e}}e^{-3\pi m|k|/2} + \frac{1}{4}e^{-2\pi m|k|}\right).$$

Furthermore, we use that $|k| \ge \frac{1}{\sqrt{2}}(|k_1| + |k_2|)$ and, therefore,

$$G(2\pi m|k|) < \pi\left(\sqrt{\frac{\pi}{e}}e^{\frac{-3\pi m(|k_1|+|k_2|)}{2\sqrt{2}}} + \frac{1}{4}e^{-\sqrt{2}\pi m(|k_1|+|k_2|)}\right).$$

Thus, summing the geometric power series, we end up with

$$F(m) < \pi - \frac{1}{m^2} + \pi\sqrt{\frac{\pi}{e}}\left(\frac{4}{(e^{\frac{3\pi}{2\sqrt{2}}m}-1)^2} + \frac{4}{e^{\frac{3\pi}{2\sqrt{2}}m}-1}\right) + \\ + \frac{\pi}{4}\left(\frac{4}{(e^{\sqrt{2}\pi m}-1)^2} + \frac{4}{e^{\sqrt{2}\pi m}-1}\right)$$

and finally show that

$$-\frac{1}{m^2} + \pi\sqrt{\frac{\pi}{e}}\left(\frac{4}{(e^{\frac{3\pi}{2\sqrt{2}}m}-1)^2} + \frac{4}{e^{\frac{3\pi}{2\sqrt{2}}m}-1}\right) + \\ + \frac{\pi}{4}\left(\frac{4}{(e^{\sqrt{2}\pi m}-1)^2} + \frac{4}{e^{\sqrt{2}\pi m}-1}\right) < 0$$

for $m \geq 1$.

A poinwise estimate

Proposition

Let for some $x \in \mathbb{R}^d$, $u(x) \in \mathbb{R}^d$ *and* $\operatorname{div} u(x) = 0$. *Then*

$$|\,((\theta, \nabla_x)u, \theta)\,(x)| \le \sqrt{\frac{d-1}{d}}\,|\theta(x)|^2 |\nabla_x u(x)|,$$

where $\nabla_x u(x)$ *is a* $d \times d$ *matrix with entries* $\partial_i u_j$, *and*

$$|\nabla_x u|^2 = \sum_{i,j=1}^{d} (\partial_i u_j)^2.$$

Thank you for your attention