Sergei Kuksin

Kolmogorov theory of turbulence and its rigorous 1d model.

Partial Differential Equations in Mathematical Physics

(conference in honour of Alexander Komech's 75th birthday)

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BASED ON:

1) my own work, starting late 1990's, on turbulence in the complex Ginzburg-Landau equation;

2) works of my former PhD students Andrey Biriuk and Alex Boritchev on turbulence in Burgers equation;

3) my book with A. Boritchev " 1d turbulence and the stochastic Burgers equation" (will appear next month in the AMS Publications).

$\S$1. K41 THEORY

The K41 theory of turbulence was created by A. N. Kolmogorov in three articles, published in 1941 (partially based on the previous work of Taylor and von Karman–Howard). It describes statistical properties of turbulent flows and is now the most popular theory of turbulence. I will refer to it for the case of a fluid flow $u(t,x)$ of order 1, space–periodic of period 1. Then the Reynolds number of the flow is

$$Rey = \nu^{-1},$$

where ν is the viscosity. If Rey is large, then the velocity field $u(t,x)$ of the flow becomes very irregular, i.e. turbulent.

All constants in my talk are independent from ν (so also from Rey).

Kolmogorov: *short scale in x features of a turbulent flow $u(t,x)$ display a universal behaviour which depends on particularities of the system only through a few parameters.*

The K41 theory is statistical. That is, it assumes that the velocity field $u(t,x)$ depends on a random parameter $\omega \in (\Omega, \mathcal{F}, \mathbf{P})$. Moreover, Kolmogorov assumes that the random field $u^\omega(t,x)$ is stationary in time and homogeneous. It means that statistically $u^\omega(t,x)$ and $u^\omega(t+C_1, x+C_2)$, where C_1, C_2 are constants, is the same random field. Kolmogorov studies short space-increments

$$u(t,x+r) - u(t,x), \quad |r| \ll 1,$$

and examines moments of these random variables as functions of $|r|$. Similar, he takes the Fourier coefficients $\hat{u}(t,s)$ of $u(t,x)$ and studies their second moments as functions of $|s|$.

The K41 theory admits a natural 1d version. I will mostly talk about it, will formulate 1d versions of the Kolmogorov claims, prove them for a 1d model, given by the Burgers equation, and then will discuss the corresponding 3d assertions of K41.

Taking the Burgers equation for a model for 1d turbulence I follow Burgers, Frisch and Sinai.

§2. THE EQUATION.

The approach I will present applies to the deterministic Burgers equation

$$u_t(t,x) + uu_x - \nu u_{xx} = 0, \qquad t \geq 0,\ x \in S^1, \quad \int u\,dx = 0,\ 0 < \nu \leq 1,$$

$$u(0,x) = u_0(x) \sim 1, \qquad 1 \leq t \leq 10.$$

Or to the stochastic equation

(B) $$u_t + uu_x - \nu u_{xx} = \partial_t \xi(t,x), \quad u(0,x) = u_0(x), \quad \int u\,dx = \int \eta\,dx = 0,$$

where $\eta = \partial_t \xi(t,x)$ and ξ is a Wiener process in the space of functions of x,

$$\xi^\omega(t,x) = \sum_{s=\pm 1, \pm 2, \dots} b_s \beta_s^\omega(t) e_s(x), \qquad B_0 := \sum_s b_s^2 < \infty.$$

Here $\{e_s, s = \pm 1, \pm 2, \dots\}$ is the trigonometric basis in the space of periodic function with zero mean, $\{\beta_s^\omega(t)\}$ are standard independent Brownian processes and $\{b_s\}$ are real numbers, fast converging to zero. So $\xi^\omega(t,x)$ is a smooth function of x. While as a function of x it is a white noise.

I like more the picture, given by the stochastic equation (B) since it is closer to Kolmogorov's insight, and will talk about it, avoiding stochastic tricks. I will understand the solutions of (B) trajectory-wise. For any ω the function $\xi^\omega(t,x)$ defines a curve

$$\xi^\omega(t,\cdot) \in C(\mathbb{R}_+, H^s), \qquad \forall\, s,$$

and $u^\omega(t,x)$ is a solution of (B) if $\forall\, \omega$,

$$u^\omega(t) - u_0 + \int_0^t (uu_x - \nu u_{xx})\, ds = \xi(t), \qquad \forall\, t \geq 0.$$

Not hard to see that if $u_0 \in H^r,\ r \geq 1$, then there is a unique solution $u^\omega \in C(\mathbb{R}_+, H^r)$ of (B). It depends on the random parameter ω, and I will systematically average in ω various functionals f of solutions u. – This is the logic of the theory of turbulence. So my goal is to study various quantities

$$\mathbf{E} f(u(t,\cdot)) = \int_\Omega f(u^\omega(t,\cdot))\, \mathbf{P}(\omega), \qquad f : H^r \to \mathbb{R}.$$

I will regard solutions $u^{\nu\omega}(t,x)$ of (B) as curves $u(t) \in L_2(S^1)$, depending on ω. I.e. as a random processes in L_2 (or in some Sobolev space).

I will soon explain that, in average, each solution $u^\omega(t,x)$ of (B) is of order one. That is,
For any u_0, $\mathbf{E}\|u(t)\|^2_{L_2} \sim 1$ uniformly in $t \geq C > 0$ and $\nu \in (0,1]$, for every $C > 0$.

Since the order of magnitude of a solution u^ν equals $\sqrt{\mathbf{E}\|u^\nu(t)\|^2_{L_2}} \sim 1$, then the solutions u are ~ 1 and their space-period is one. Thus the Reynolds number of each u is Rey$(u^\nu) \sim \nu^{-1}$. So (B) with small ν describes 1d turbulence, called by Uriel Frisch *burgulence*.

THE GOALS are: 1) to study the solutions $u^{\nu\omega}(t,x)$ for small ν and for $0 < t \leq \infty$,
2) to relate the obtained results with the K41 theory, regarding the Burgers equation (B) as a 1d model for turbulence.

Inspired by the heuristic work on the stochastic Burgers equation by U. Frisch with collaborators, Sinai and others in the paper

E, Khanin, Mazel, Sinai *Invariant measures for Burgers equation with stochastic forcing*, Ann. Math. **151**, 877-960 (2000)

used the Lax-Oleinik formula to write down the limiting dynamics of (B) as $\nu \to 0$, and next studied the obtained limiting random field $u(t,x)$. In this way they arrived at a beautiful theory and solved the problem 1) above, but their solution does not allow to obtain the relations, claimed by the theory of turbulence.

On the contrary, we study (B) for small but POSITIVE ν, i.e. when

not $\nu \to 0$, but $0 < \nu \ll 1$,

using basic tools from PDEs and stochastic processes. This approach allows to get relations, similar to those, claimed by the K41 theory. Our results rigorously justify the heuristic theory of 1d turbulence, built in the paper

Aurell, Frisch, Lutsko, Vergassola, J. of Fluid Mechanics, **238**, 467–486, 1992.

§3. APRIORY ESTIMATES (= upper bounds for the norms).

The key starting point is the Oleinik-Kruzkov inequality, which we apply to solutions of (B) with fixed ω. The inequality was proved by O–K for the free Burgers equation, but their argument applies to the stochastic equation (B) trajectory-wise, and implies the following:

THEOREM O-K. For ANY initial data u_0, any $p \geq 1$ and any $\nu, \theta \in (0,1]$, uniformly in $t \geq \theta$ we have:

(O-K) $$\mathbf{E}\big(|u^\nu(t,\cdot)|_\infty^p + |u_x^\nu(t,\cdot)|_1^p\big) \leq C\theta^{-p}.$$

The constant C depends only on the random force (NOT on ν and NOT on u_0 !).

Idea of the proof: If ξ is zero, consider $w = tu_x$, find a point where it takes maximal value at the cylinder $S^1 \times [0,T]$, and write there the condition of maximality. If ξ is not zero, take $v = u - \xi$, denote $w = tv_x$, and do the same.

This very powerful estimate, jointly with some PDE tricks, allows to bound from above moments of all Sobolev norms of solutions:

THEOREM 1. For any u_0, every $m \in \mathbb{N}$, $0 < \nu \leq 1$ and every $\theta > 0$,

$$\mathbf{E}\|u^\nu(t)\|_m^2 \leq C(m,\theta)\nu^{-(2m-1)} \quad \text{if} \quad t \geq \theta.$$

Remark. For $m = 0$ this is wrong, and instead then we have $\mathbf{E}\|u^\nu(t)\|_0^2 \sim 1$.

§4. LOWER BOUNDS FOR MOMENTS OF SOBOLEV NORMS OF SOLUTIONS.

For the stochastic equation (B) the Balance of Energy Relation for solutions takes the following nice form:

$$\mathbb{E}\int \tfrac{1}{2}|u(T+\sigma,x)|^2dx-\mathbb{E}\int \tfrac{1}{2}|u(T,x)|^2dx+\nu\mathbb{E}\int_T^{T+\sigma}\int |u_x(s,x)|^2dxds=\sigma B_0,$$

where $T,\sigma>0$ and $B_0=\sum b_s^2>0$.

Let $T\geq 1$. Then by Oleinik-Kruzkov, the first two terms are bounded by a constant C_*, which depends only on the random force. If $\sigma\geq\sigma_*=4C_*/B_0$, then $C_*\leq\frac{1}{4}\sigma B_0$ and we get that

$$\boxed{\nu\mathbb{E}\,\frac{1}{\sigma}\int_T^{T+\sigma}\int |u_x^\nu(s,x)|^2dxds\geq\tfrac{1}{2}B_0.}$$

For any random function $f^\omega(t)$ (i.e. for a random process f) I will denote by $\langle\langle f\rangle\rangle$ its averaging in ensemble and local averaging in time,

$$\langle\langle f\rangle\rangle = \mathbb{E}\,\frac{1}{\sigma}\int_T^{T+\sigma} f(s)\,ds, \qquad \text{where } T\geq 1,\ \sigma\geq\sigma_* \text{ are parameters.}$$

In this notation we have just proved that

$$\langle\langle \|u^\nu\|_1^2\rangle\rangle \geq \nu^{-1}\,\tfrac{1}{2}B_0.$$

But by Theorem 1 $\langle\langle \|u^\nu\|_1^2\rangle\rangle \leq \nu^{-1}C$! So

$$\langle\langle |u_x^\nu|_{L_2}^2\rangle\rangle = \langle\langle \|u^\nu\|_1^2\rangle\rangle \sim \nu^{-1},$$

where $\sim$ means that the ratio of two quantitis is bounded from below and from above, uniformly in ν and in $T\geq 1$ and $\sigma\geq\sigma_*$, entering the definition of the brackets $\langle\langle\cdot\rangle\rangle$.

Now the Gagliardo-Nirenberg interpolation inequality + Oleinik-Kruzkov imply:

$$\langle\langle |u_x^\nu|_{L_2}^2\rangle\rangle \overset{G-N}{\leq} C'_m \langle\langle \|u^\nu\|_m^2\rangle\rangle^{\frac{1}{2m-1}} \langle\langle |u_x^\nu|_{L_1}^2\rangle\rangle^{\frac{2m-2}{2m-1}} \overset{O-K}{\leq} C_m \langle\langle \|u^\nu\|_m^2\rangle\rangle^{\frac{1}{2m-1}}$$

Using the already obtained lower bound for the averaged first Sobolev norm, $\langle\langle \|u^\nu\|_1^2\rangle\rangle \geq \nu^{-1}B_0/2$, we get from here a lower bound for $\|u^\nu\|_m$:

$$\langle\langle \|u^\nu\|_m^2\rangle\rangle \geq C''_m \nu^{-(2m-1)} \qquad \forall m \geq 1.$$

Combining this with the upper bound in Theorem 1, we get:

THEOREM 2 (*Sobolev norms of solutions*). For any u_0, any $0 < \nu \leq 1$ and every $m \in \mathbb{N}$

,

(Basic Estimate) $$\langle\langle \|u^\nu\|_m^2\rangle\rangle \sim \nu^{-(2m-1)}.$$

This theorem and the Oleinik–Kruzkov result turns out to be a powerful and efficient tool to study the turbulence in the 1d Burgers equation (B) (the burgulence).

Open problem. Prove (or disprove) that $\langle\langle \|u^\nu\|_m^2\rangle\rangle$ admits an asymptotic expansion:

$$\langle\langle \|u^\nu\|_m^2\rangle\rangle = C_m \nu^{-(2m-1)} + o(\nu^{-(2m-1)}).$$

$\S$5. Burgulence: the dissipation scale

I recall that I write $u(t,x)$ as Fourier series $\sum_{s=\pm1,\pm2,\dots} \hat{u}_s(t) e_s(x)$ in the trig. basis.

By a direct analogy with K41, the basic quantity characterising a solution $u(t,x)$ as a 1d turbulent flow is its *dissipation scale* l_d, a.k.a. *Kolmogorov's inner scale*. We define it in the Fourier presentation as :

l_d is the smallest number of the form $l_d = \nu^{-c_d}$, $c_d > 0$, such that for $|s| \gg l_d$ the avaraged squared norm of the s-sth Fourier coefficient $\hat{u}_s(t)$ decays very fast.

Namely, for any $N \in \mathbb{N}$ and $\gamma > 0$ there exists a $C_{N,\gamma}$ such that

$$\langle\langle |\hat{u}_s(t)|^2 \rangle\rangle \le C_{N,\gamma} |s|^{-N}, \quad \forall\, |s| \ge \nu^{-c_d-\gamma},$$

Theorem 2 and a Tauberian argument imply:

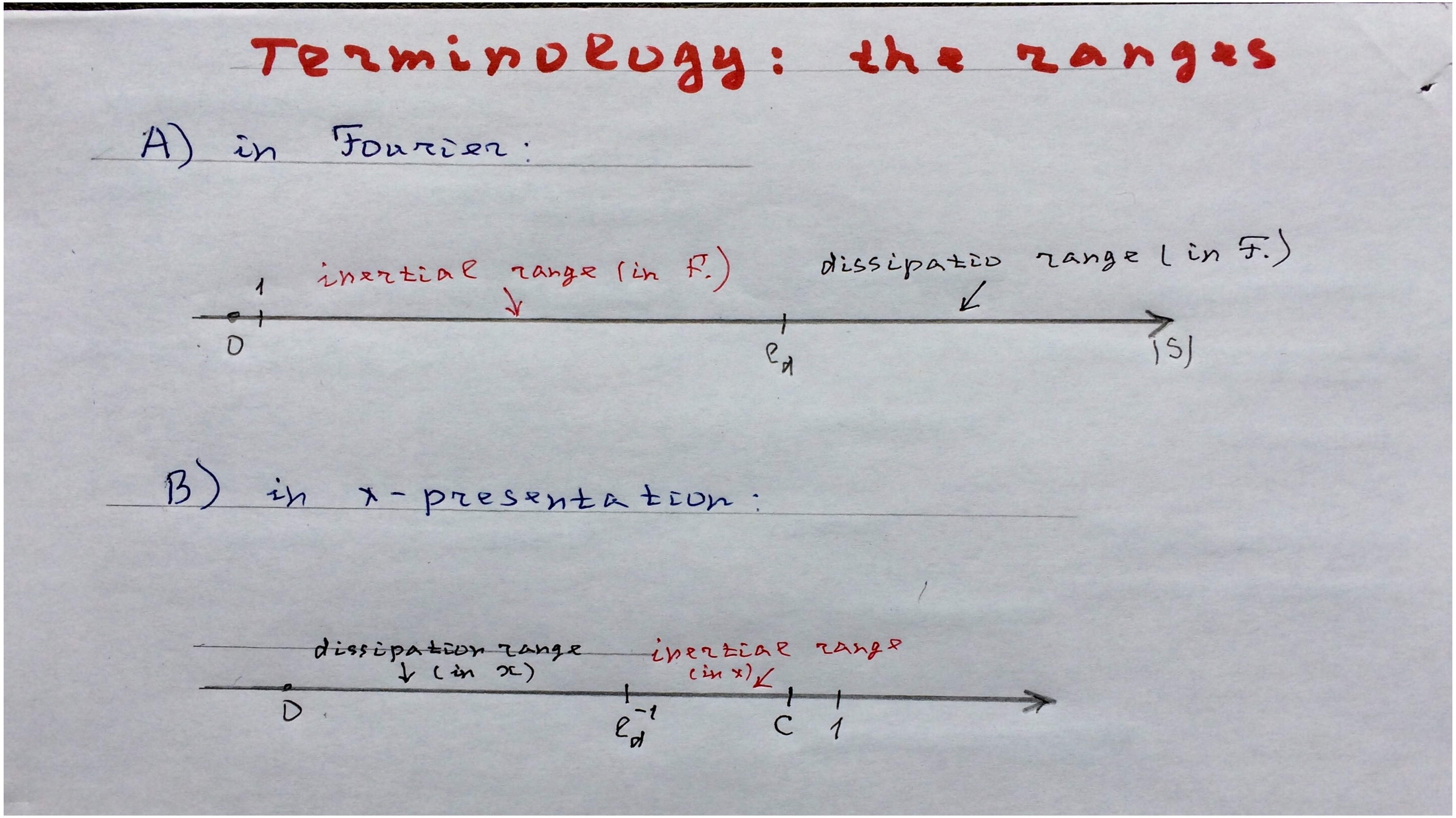
THEOREM 3. The dissipation space-scale l_d of any solution u of (B) equals ν^{-1}.

i) In physics, the dissipative scale l_d is defined modulo a constant factor, so for Burgers $l_d =$Const ν^{-1}.

ii) In K41 the hydrodynamical dissipative scale is predicted to be $l_d^K = \nu^{-3/4}$.

iii) For the Burgers equation, Burgers hinself predicted the correct dissipative scale $l_d = \nu^{-1}$.

In the theory of turbulence ranges are zones, specifying the size of the involved Fourier modes s and of increments of x:



Recall that for (B), $l_d = \text{Const}\,\nu^{-1}$. Here $C > 0$ depends only on the random force.

§6. Burgulence: moments of small-scale increments.

Small-scale increments: $|u(x+l)-u(x)|, \quad x \in S^1, \quad |l| \ll 1.$

Their moments are

$$\langle\langle |u(x+l)-u(x)|^p\rangle\rangle =: S_p(l;u), \quad p>0.$$

Function $(l,p) \mapsto S$ is called *the structure function* (corresponding to a solution u). Recall that inertial range in x is $[l_d^{-1}, C] = [C_1\nu, c]$, and dissipation range in x is $(0, l_d^{-1}] = (0, C_1\nu]$. The function S obeys the following law:

THEOREM 4. For $|l|$ in the inertial range $[C_1\nu, c]$ we have

(1)
$$S_p(l;u^\nu) \sim |l|^p \quad \text{if} \quad 0<p\le 1,$$
$$S_p(l;u^\nu) \sim |l| \quad \text{if} \quad p \ge 1.$$

While for $|l|$ in the dissipation range $[0, C_1\nu]$,

(2)
$$S_p(l,u^\nu) \sim |l|^p \nu^{1-\min(p,1)}, \quad \forall\, p>0.$$

$$\langle\langle |u(x+l) - u(x)|^p \rangle\rangle =: S_p(l;u), \quad p > 0.$$

U.Frisch with collaborators obtained the first assertion in the theorem above :

(1) $$S_p(l;u^\nu) \sim |l|^{\min(1,p)} \quad \text{for} \quad p > 0, \quad |l| \in [c_1\nu, c],$$

by a convincing heuristic argument in

Aurell, Frisch, Lutsko, Vergassola, J. of Fluid Mechanics, **238**, 467–486, 1992.

We rigorously derive (1) and (2) from Theorem 2 (*Sobolev norms of solutions*) and Oleinik-Kruzkov, using some ideas from the paper above.

For water turbulence the K41 theory predicts that in the inertial range we have

(1/3 law) $$S_p(l) \sim |l|^{p/3}, \qquad |l| \in [C\nu^{3/4}, C_1].$$

This is the celebrated *1/3 law of the K41 theory*. It claims that – in a sense – the sizes of increments $|u(x+l) - u(x)|$ behaves as $|l|^{1/3}$ for $|l|$ in the inertial range .

From the point of view of K41, the obtained law (1) presents an ”abnormal scaling”.

For which p does the 1/3-law holds?

DISCUSSION. In the Kolmogorov setting u is a homogeneous random field, and

$$S_p(l) = \mathbf{E}|u(x+l) - u(x)|^p$$

– no need to average in x and in t This is the p-th moment of the random variable $u(x+l) - u(x)$. The 1/3-law tells us that

$$\frac{S_p(l)^{1/p}}{S_q(l)^{1/q}} \sim C_{p,q} \qquad \forall\, p, q > 0,$$

even for tiny $|l|$, when the random variable $u(x+l) - u(x)$ is very small. Relation above would hold, with absolute constants $C_{p,q}$, if $u(x+l) - u(x)$ was a Gaussian random variable. But $u(x+l) - u(x)$ certainly is not Gaussian - this is suspicious!

In our case, for burgulence,

$$\frac{S_p(l)^{1/p}}{S_q(l)^{1/q}} \sim C_{p,q}\, |l|^{1/p-1/q},$$

which is big for small l if $p > q$. This is a "very non-Gaussian behaviour" (the function above with $p = 4,\ q = 2$ is called *flatness* of the random variable $u(x+l) - u(x)$).

§7. Burgulence: distribution of the energy along spectrum.

The second celebrated law of the Kolmogorov theory deals with the distribution of the energy $\langle\langle \frac{1}{2}\int |u|^2 dx\rangle\rangle$ along the spectrum. For a solution $u(t,x)$, regarded as a 1d turbulent flow, consider $\frac{1}{2}\langle\langle |\hat{u}_s|^2\rangle\rangle$. By Parseval's identity,

$$\langle\langle \frac{1}{2}\int |u|^2 dx\rangle\rangle = \sum_s \frac{1}{2}\langle\langle |\hat{u}_s|^2\rangle\rangle.$$

So the quantities $\frac{1}{2}\langle\langle |\hat{u}_s|^2\rangle\rangle$ characterise distribution of the energy along the spectrum.

Next, for any $k \in \mathbb{N}$ define $E_k(u)$ as the averaging of $\frac{1}{2}\langle\langle |\hat{u}_s|^2\rangle\rangle$ along the layer J_k around k, $J_k = \{n \in \mathbb{Z}^* : M^{-1}k \le |n| \le Mk\}$, i.e.

$$E_k(u) = \langle\langle e_k(u)\rangle\rangle, \quad e_k(u) = \frac{1}{|J_k|}\sum_{n\in J_k} \frac{1}{2}|\hat{u}_n|^2;$$

$e_k(u)$ is the averaged energy of the k-th mode of u. The function $k \mapsto E_k$ is called the *energy spectrum*. It is immediate that for $k \gg l_d$ it decays faster than any negative degree of k (uniformly in ν). But for $k \le l_d$ the behaviour of E_k is quite different:

Theorem on the Structure Function S and a Tauberian argument imply the Spectral Power Law:

THEOREM 5. For k in the inertial range, $1 \le k \le C_1 \nu^{-1}$, we have:

(Spectral Power Law) $$E_k(u^\nu) \sim k^{-2}.$$

For the water turbulence the K41 theory predicts that E_k obeys the celebrated Kolmogorov–Obukhov law:

(K-O law) $$E_k \sim |k|^{-5/3} \quad \text{for } k \text{ in the inertial range.}$$

For solutions of (B), Jan Burgers in 1940 (!) predicted that $E_k \sim |k|^{-2}$ for $|k| > \text{Const}\, \nu^{-1}$, i.e. exactly the Spectral Power Law above.

§8. THE MIXING. The mixing means that in the function space H of functions of x, where we study the equation, there exists a unique measure μ_ν, such that for any "reasonable" functional f on H and for any solution $u^{\nu\omega}(t,x)$ of (B) we have

$$\mathbb{E} f(u(t,\cdot)) \to \int_H f(u)\, \mu_\nu(du) \quad \text{as} \quad t \to \infty.$$

In physics, μ_ν is called a *statistical equilibrium* for eq. (B).

This holds for (B), and may be derived from a general theory. But then the rate of convergence would depend on ν. In the same time, in the theory of turbulence it should not depend on ν, and for (B) it does not!

THEOREM 6. If the functional $f(u)$ is continuous in some L_p–norm, $p < \infty$, then the rate of convergence above does not depend on ν, and holds at least with the rate $(\ln t)^{-\kappa_p}$.

Recall that $\xi^\omega(t,x) = \sum_{s=\pm1,\pm2,\dots} b_s \beta_s^\omega(t) e_s(x)$. If $b_s \equiv b_{-s}$, then the random field $\xi(t,x)$ is homogeneous in x. In this case the measure μ_ν also is homogeneous.

Stationary solution $u^{st}(t)$ of (B) is a solution such that $\mathcal{D}u^{st}(t) = \mu_\nu$ for all t.

Energy spectrum of the stationary measure μ_ν is

$$E_k(\mu_\nu) = \int e_k(u)\mu_\mu(du), \qquad e_k(u) = \frac{1}{|J_k|}\sum_{n\in J_k} \tfrac{1}{2}|\hat{u}_n|^2.$$

Obviously,

$$E_k(\mu_\nu) = \langle\langle e_k(u^{st}(t))\rangle\rangle = \mathbf{E}e_k(u^{st}(t)).$$

Since $\langle\langle e_k(u^{st}(t))\rangle\rangle$ satisfies the Spectral Power Law, then $E_k(\mu_\nu)$ also does:

$$E_k(\mu_\nu) \sim k^{-2}.$$

Equivalently,

$$\mathbf{E}e_k(u^{st}(t)) \sim k^{-2} \text{ for all } t.$$

This is in the spirit of K41, where the velocity fields $u(t,x)$ are assumed to be stationary in t (and homogeneius in x).

If the force ξ is such that $b_s \equiv b_{-s}$, then $u^{st}(t,x)$ is stationary in t and homogeneous in x. – A perfect match for K41!

§9. THE INVISCID LIMIT. Another remarkable feature of (B) is that, as $\nu \to 0$, a solution u^ν of (B) converges to an inviscid limit:

$$u^\nu(t,\cdot) \to u^0(t,\cdot) \quad \text{in } \; L_p(S^1), \; \text{a.s.},$$

for each $p < \infty$. This result is due to Lax–Oleinik (1957). The limit $u^0(t,x)$ is called an "inviscid solution", or an "entropy solution" of (B) with $\nu = 0$. The limiting function $u^0(t,x)$ even is not continuous. But the structure function and spectral energy are well defined for L_p–smooth random fields $U(x)$, so they are well defined for $u^0(t,x)$. It turns out that they inherit the laws, proved for u^ν with $\nu > 0$:

THEOREM 7. For all $\nu > 0$,

1) $E_k(u^0) \sim k^{-2}$ for all k;

2) $S_p(l, u^0) \sim |l|^p$ if $0 < p \le 1, \;\; |l| \le c$, and
$S_p(l, u^0) \sim |l|$ if $p \ge 1, \;\; |l| \le c$.

Now the energy law for E_k holds for ALL $k \ge 1$. So for u^0 the dissipation scale l_d equals ∞. The inertial range in Fourier becomes all of $\mathbb{N}$, and in x it is the whole interval $(0, c]$. This theorem describes the *inviscid burgulence*. It has no analogy in K41.

CONCLUSIONS. The stochastic Burgers equation with small viscosity makes a consistent model of 1d turbulence. Its rigorously proved statistical properties make natural and close analogies for the main laws of the K41 theory

of turbulence. This, once again, supports the belief that the K41 theory is “close to the truth”.