Conference in honor of Alexander Komech's 75th birthday

Soliton asymptotics in hydrodynamics

Alexander Shnirelman

1. Soliton is a localized stable structure appearing in a solution of certain nonlinear equations. For some equations (like KdV, NLS, Maxwell-Schroedinger, ...) generic solution behaves at $t \to \infty$ as a collection of independent solitons. Such collections of finite or countable number of solitons form a (weak) attractor of such systems. Many works of Alexander Komech are devoted to the soliton asymptotics.

2. <u>2-dimensional hydrodynamics</u>

Consider the motion of 2-d ideal incompressible fluid in a compart 2-d domain M (or 2-d torus $\boldsymbol{T}^2$). It is described by the Euler equations

$$\frac{\partial u}{\partial t} + \nabla u \cdot u + \nabla p = 0 \ ;$$
$$\nabla \cdot u = 0 \ ;$$
$$u_n|_{\partial M} = 0.$$

Let $\omega = \nabla \times u = \mathrm{curl}\, u$; then the vorticity equation holds:

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = 0 \ ,$$
$$u = \mathrm{curl}^{-1} \omega \ .$$

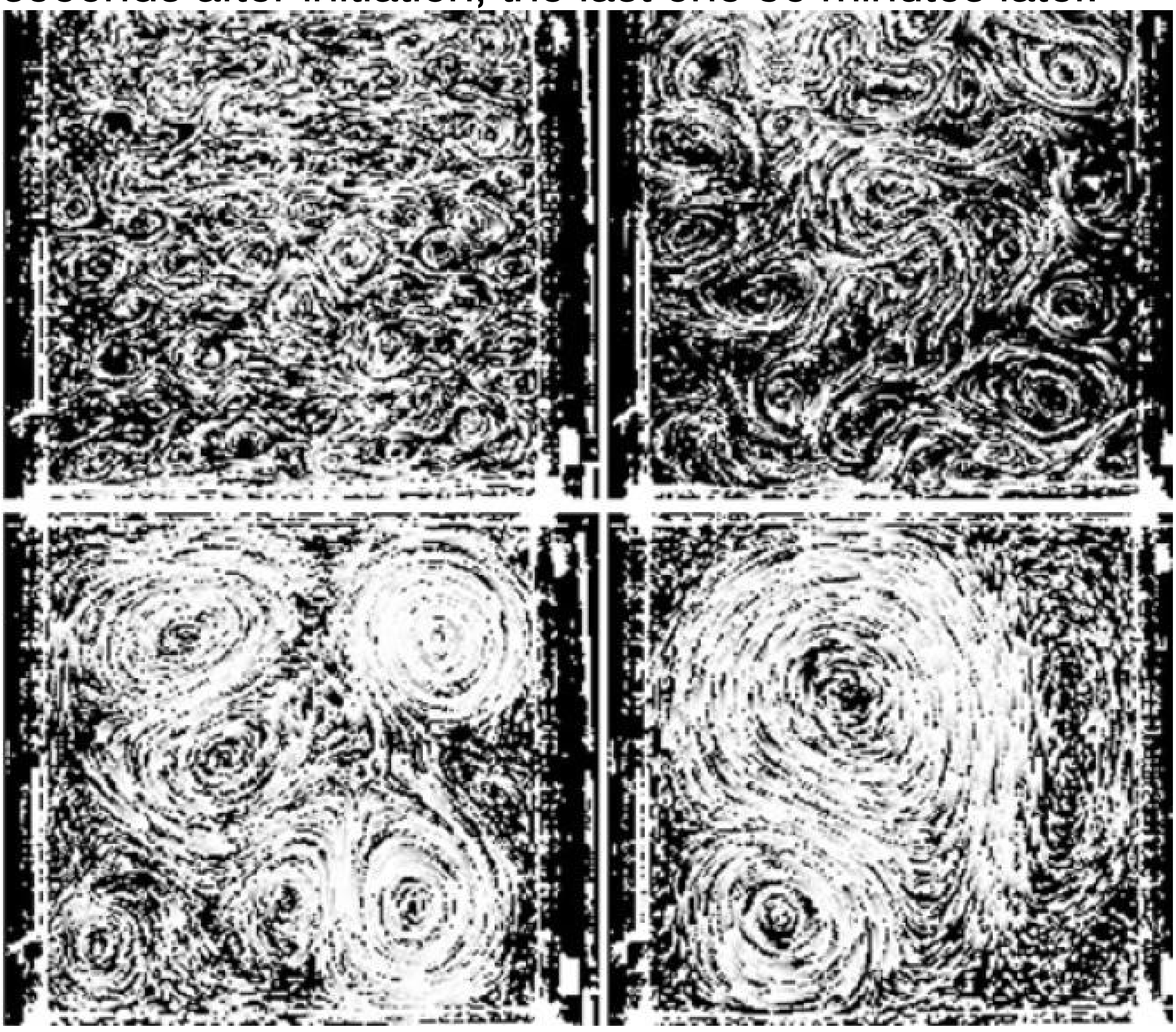
The Lagrangian description: $(x, t) \rightarrow g_t(x)$ - flow map.

<u>Vorticity theorem:</u>
$\omega(x, t) = \omega_0\left(g_t^{-1}(x)\right)$ where $\omega_0 = \omega|_{t=0}$.

3. Long-time behavior of 2-d ideal flows

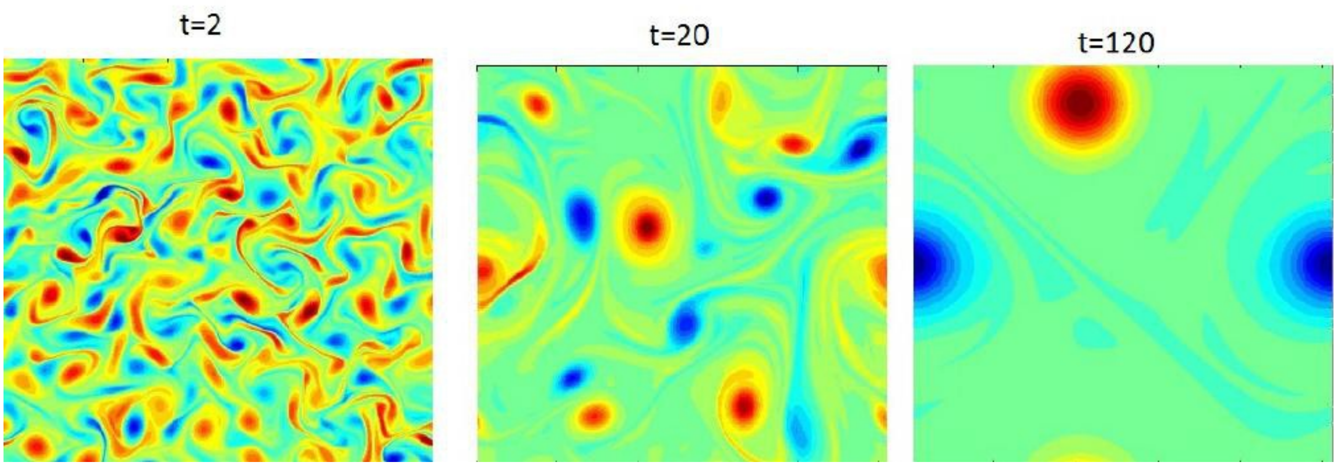
Our intuition is not strong enough to guess how the fluid behaves in a long run. So, we should turn to experiments.

(a) Physical experiment (Clerx et al., 2011). A thin mesh was dragged through a shallow (3 cm) layer of water in a square tray 1m $\times$1m; the first picture is taken 10 seconds after initiation, the last one 50 minutes later.



(b) Numerical simulation.

Euler equations on the unit torus were solved by pseudospectral method combined with 4-th order Runge-Kutta method in time. The initial vorticity was a linear combination of plane waves with the wavelength $\lambda \sim 0.05$ Here is the picture of vorticity at indicated time moments.



We see that in both cases the initial configuration of several hundred of small vortices degenerates to just a couple of large vortices through the cascade process of successive mergers. The final configuration of just a few large vortices is robust and stable; it is the fluid analogue of a soliton.

Our goal is to explain this result from the first principles.

4. Early (and, possibly, wrong) theories.

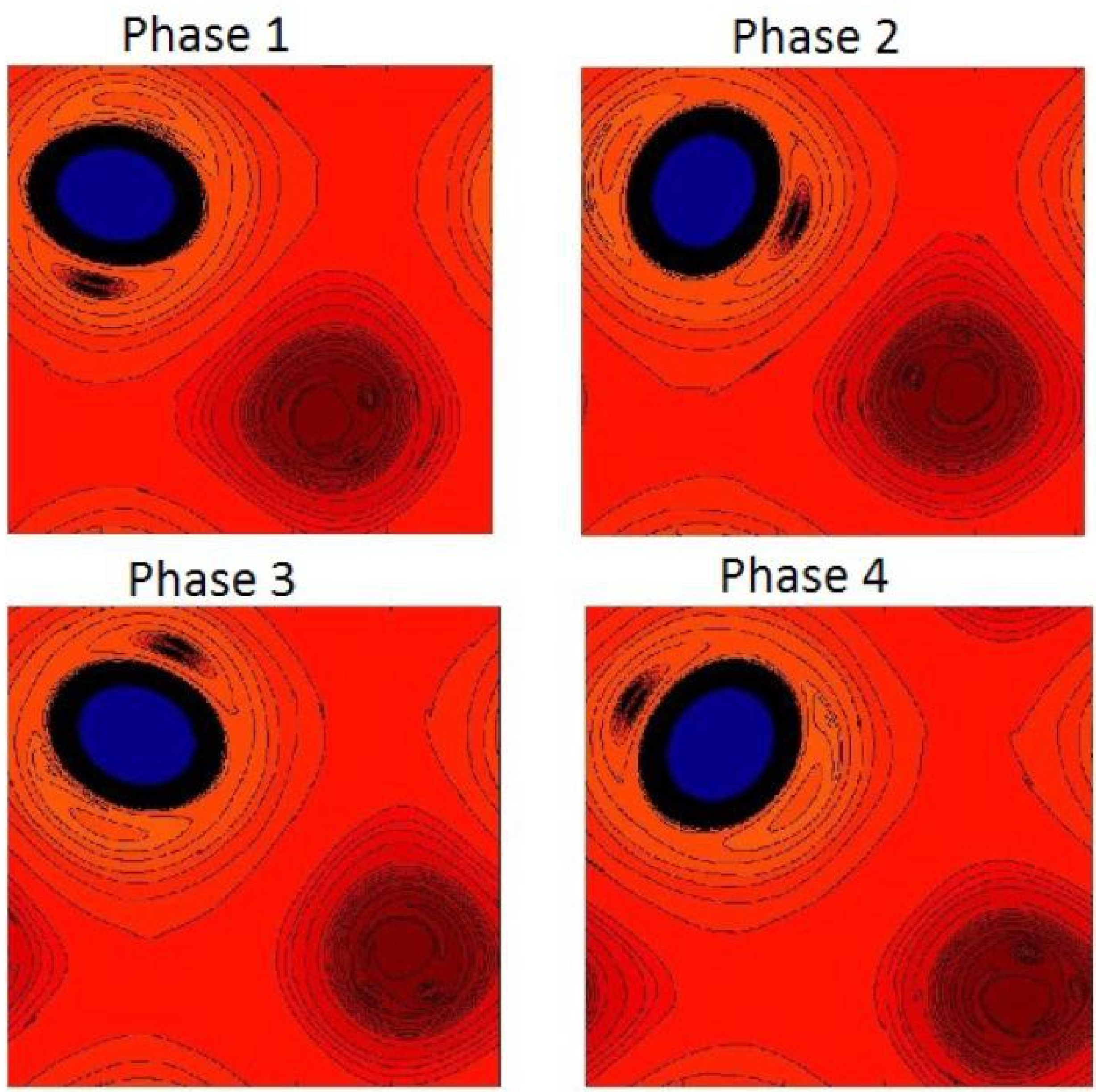
(a) "Statistical theory" (Miller-Robert, Sommeria, ...) The system is approximated by a finite-dimensional, or even a finite, discrete-time system of size (dimension, number of states) N , and a microcanonical measure μ is defined. As $N \to \infty$, the measure is concentrated near a single state $\bar{u}$ (by the large deviations theorem). This state depends on the energy and, possibly, some moments of vorticity of the initial velocity u_0 , and is a stationary of the Euler equations. The basic conjecture is that this is the final state of the flow.

(b) Mixing theory (Shn.) The vorticity ω is transported by the flow and, in a long run, it is irreversibly mixed, while the kinetic energy is conserved. The main conjecture is that the mixing goes on until the further mixing becomes impossible, i.e. any further mixing changes the energy. For any initial vorticity ω_0 such "maximally mixed" state (I call it "minimal flow") is a stationary solution of the Euler equations; in general, it is not unique. I conjectured that one of the minimal flows is the final state.

Both theories implied that any solution tends to some stationary solution as $t \to \infty$. But this conclusion turned out to be wrong.

5. <u>Accurate numerical simulations</u> show that the flow tends to some asymptotic regime, which may be not a stationary flow.

Here is a periodic solution which is a result of a pretty long evolution.



In this flow the sets $S_c(t)=\{x \mid \omega(x,t)\leq c\}$ are transported by the flow, but they don't mix: every component of $S_c(t)$ keeps its individuality. Those "nonmixing" sets have a complicated, hierarchical structure; they form "islands", "archipelagos", "lakes", etc, and inside an island there may be seen smaller islands moving around, and the number of such nested islands may be arbitrarily large.

Let us give a formal definition.

6. Generalized minimal flows (GMF)

Let V denote the Yudovich space of incompressible vector fields in M such that $\operatorname{div} u=0$, and $\operatorname{curl} u\in L^\infty$. Let $S_t\colon V\to V$ be the 1-parameter group of transformations of V generated by the Euler equations. For any $u\in V$, let $\mathcal{O}(u) = S_{\boldsymbol{R}+}u$ be the semi-orbit of u. Let $[\mathcal{O}(u)]$ denote the set of limit points of $\mathcal{O}(u)$ in L^2.

Definition. *A vector field $u\in V$ is called a generalized minimal flow (GMF) if for any $v\in[\mathcal{O}(u)]$,*
$\|\operatorname{curl} v\|_{L^2}=\|\operatorname{curl} u\|_{L^2}$.

Let $\mathcal{N}$ denote the set of all such u.

<u>Conjecture.</u> *The set* $\mathcal{N}$ *is a global* L^2 *- attractor in the space* V *.*

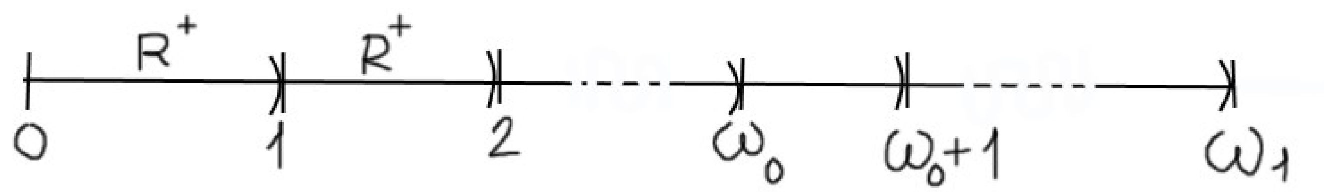
This conjecture is true if we replace the dynamics in V by some its modification.

7. The long line (Alexandroff's line) as a replacement of the time axis

Ordinal is an equivalence class of well ordered sets. If k, n are two ordinals, then either k is an initial segment of n, or n is an initial segment of k, or $k=n$.

Let ω_1 be the smallest uncountable ordinal; then $\omega_1 = \bigcup_{k \text{ is countable}} k$.

Alexandroff line $AL = \omega_1 \times \boldsymbol{R}^+$; it is endowed by the lexicographic order and the order topology:



Elements of AL are denoted by $\tau = (n, t)$, $n \in \omega_1$, $t \in \boldsymbol{R}^+$.

8. Pseudoevolution

Let $u_0 \in V$; let $S_t u_0$ be the solution of the Euler equations with the initial velocity u_0 . For any $t \in \boldsymbol{R}^+$ we define $u(0,t) = S_t u_0$ where $(0,t)$ is regarded as an element of AL . If $u_0 \in \mathcal{N}$, we define u_1 as any element of $[\mathcal{O}(u_0)]$. Otherwise we choose $u_1 \in [\mathcal{O}(u_0)]$ such that $\|\mathrm{curl}\, u_1\|_{L^2} < \|\mathrm{curl}\, u_0\|_{L^2}$. Then we define $u(1,t) = S_t u_1$. Then we define in the same way $u_2, u_3, \ldots$. The field u_{ω_0} is defined as any L^2 - limit element of the sequence $u_0, u_1, \ldots$. At the same time we define $u(n,t)$ for all $n \in \omega_0$. Then we define $u_{\omega_0+1}, u_{\omega_0+2}, \ldots$, and, by the transfinite recursion, we define u_n and $u(n,t)$ for all $n \in \omega_1$, $t \in \boldsymbol{R}^+$. The sequence $\{a_n = \|\mathrm{curl}\, u_n\|_{L^2}\}$ is defined for all countable ordinals $n < \omega_1$.

9. Lemma: *Let* $\{a_n\}$ *be a monotone non-increasing sequence of real numbers defined for all* $n<\omega_1$ *. Then there exists an index* $k<\omega_1$ *such that* $a_n=a_k$ *for all* $n>k$ *. In other words, the sequence* $\{a_n\}$ *stabilizes after some* $k<\omega_0$ *.*

Hence, $u_k=u(k,0)\in\mathcal{N}$, i.e. u_k is a Generalized Minimal Flow.

Our solution $u(\tau)=u(n,t)$ undergoes an infinitesimal jump, or a "clinamen" (Epicurus, Lucretius) at $\tau=(n,0)$ for every $n<\omega_1$; hence the term "pseudoevolution". This means that the set $\mathcal{N}$ is actually achieved at some moment $\tau\in AL$ (in a sense, Achilles catches up with a tortoise).

This is not a proof that $\mathcal{N}$ is an L^2 - attractor in a usual sense. The question if $\mathcal{N}$ is a "true" attractor remains open.

10. Generalized Minimal Flows and Landau damping

Is $\mathcal{N}$ nontrivial, i.e. s.t. (1) $\mathcal{N} \neq \emptyset$, and (2) $\mathcal{N} \neq V$?

(1) $\mathcal{N} \neq \emptyset$ because there exist a plenty of stationary flows.

(2) J. Bedrosian and N. Masmoudi proved (2013) that there exists a smooth perturbation $u(x, t)$ of the Couette flow $u_0(x_1, x_2) = (x_2, 0)$ in the concompact domain $M = T \times \boldsymbol{R}$ which is not a parallel flow, but $\lim_{t \to \infty} u(x, t)$ is a parallel flow, and $\operatorname{curl} u$ becomes more and more oscillating as $t \to \infty$. This would mean that $u \notin \mathcal{N}$, but unfortunately their domain M in not compact. Example in a compact domain is still to be constructed.

11. The entropy problem

Soliton asymptotics should make a physicist feel uneasy. In fact, they mean that in a conservative physical system the diversity of outcomes appears much smaller than the diversity of the initial conditions. If the initial condition in the corresponding metric space is random with probability distribution μ_0, then the above statement means that the "entropy" of μ_0 is much larger than that of μ_t as $t \to \infty$.
(Here by the "entropy" I mean any measure of the diversity; for example, it may be the $\varepsilon-\delta$ entropy

$$H_{\varepsilon,\delta}(\mu) = \inf\{H_\varepsilon(A) \mid \mu(A) > 1-\delta\}$$

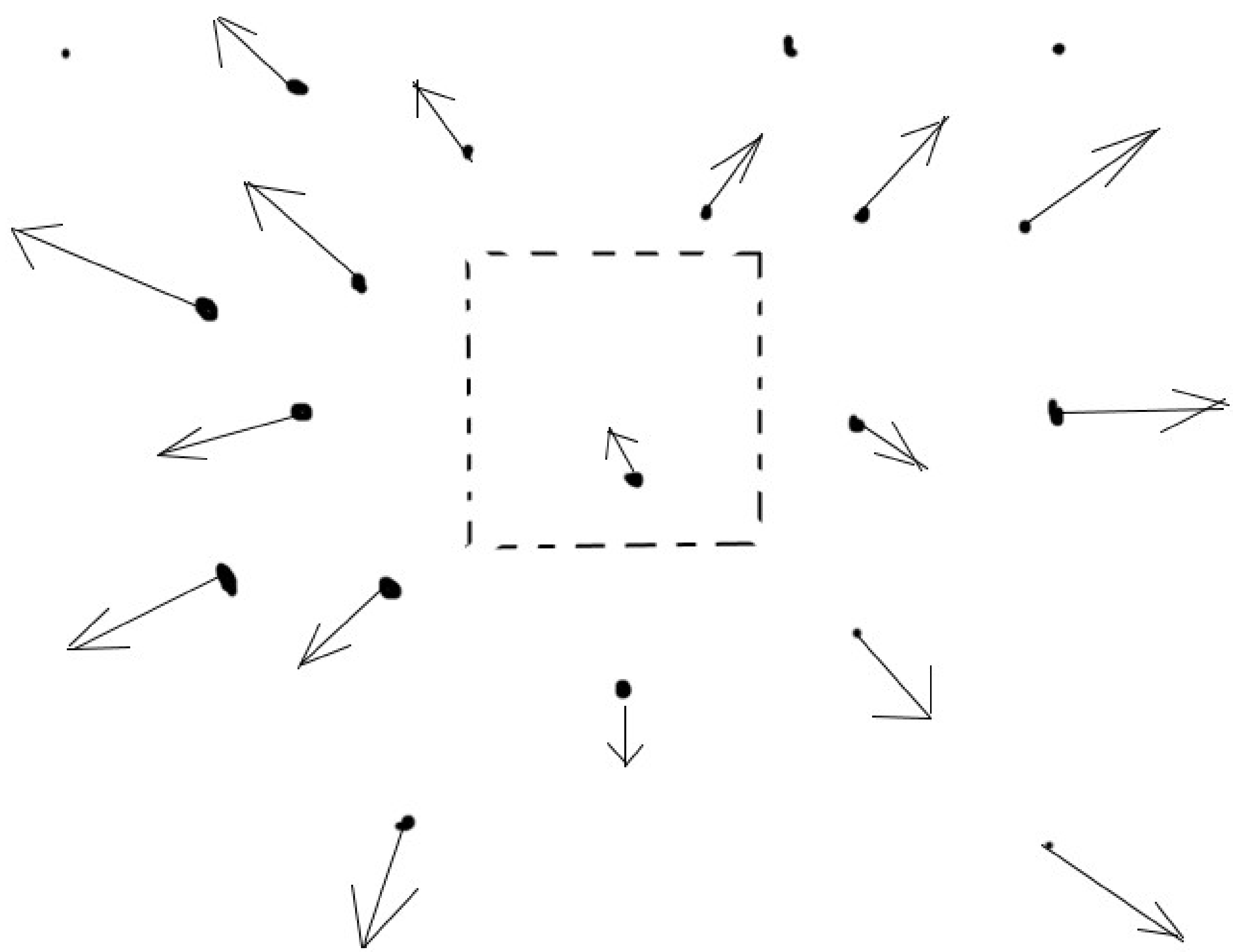
where H_ε is the Kolmogorov's ε - entropy).

But in a conservative system the entropy cannot decrease! Where does it go?

12. The answer: Every time we see the entropy decreases, we observe only a part of of the variables describing the system. There exist other variables, whose entropy increases, so that the total entropy does not decrease.

Example: expansion of an ideal gas in the empty space

Suppose at some moment we have released some mass of an ideal gas in the empty space, and since that the gas is freely expanding.



Then the temperature of expanding gaz (in the comoving frame) would decrease, together with the entropy of the molecules' velocities (which is a purely kinematic effect). However, the volume of the gas increases, together with the entropy of the molecules' positions. As a result, the total entropy does not decrease. The entropy is flowing from the momentum to the position space.

13. <u>But where is the second half of variables describing the fluid?</u>

The fluid state is described by positions and velocities of all its particles: $z=(g(x), V(x))$ where $x \in M$ is the particle label, $g \in SDiff(M)$, the group of volume preserving diffeomorphisms, and the velocity field $u(x) = V(g^{-1}(x))$.

<u>Theorem</u> (Y. Eliashberg, T. Ratiu, 1992). *If* $dim(M)=2$ *then diameter (in the* L^2 *metric) of the group* $SDiff(M)$ *is infinite.*

Hence there is enough space to absorb any amount of entropy.

Sasha, my best wishes to you!