Control of eigenfunctions on negatively curved surfaces

Semyon Dyatlov (MIT)

Feb 27, 2021

- This talk presents a recent result in quantum chaos
- Central ingredient: fractal uncertainty principle (FUP)

No function can be localized
in both position and frequency
near a fractal set

- Using tools from
  - Microlocal analysis (classical/quantum correspondence)
  - Hyperbolic dynamics (classical chaos)
  - Fractal geometry
  - Harmonic analysis

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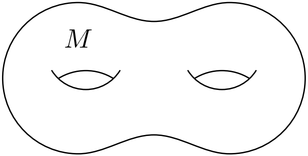
Control of eigenfunctions

- (M, g) negatively curved surface
- Geodesic flow $\varphi_t : T^*M \to T^*M$ is a standard model of classical chaos
- Eigenfunctions of the Laplacian $-\Delta_g$ studied by quantum chaos



$$(-\Delta_g - \lambda^2)u = 0, \quad \|u\|_{L^2} = 1$$

Theorem 1

Let $\Omega \subset M$ be an arbitrary nonempty open set. Then

$$\|u\|_{L^2(\Omega)} \geq c > 0$$

where c depends on M, Ω but not on λ

Constant curvature: D–Jin '18, using D–Zahl '16 and Bourgain–D '18

Variable curvature: D–Jin–Nonnenmacher '19, using Bourgain–D '18

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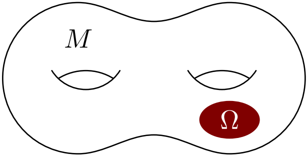

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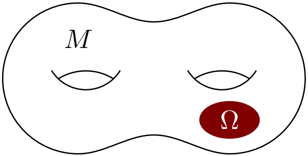

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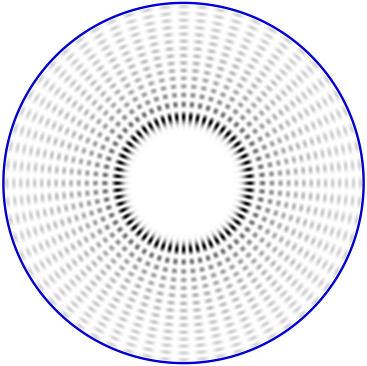
For bounded λ the estimate follows from unique continuation principle

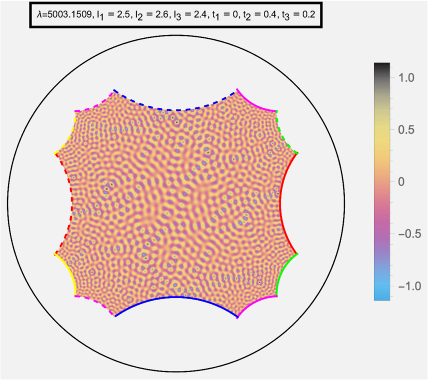
The new result is in the high frequency limit $\lambda \to \infty$

An illustration

Picture on the right courtesy of Alex Strohmaier, using Strohmaier–Uski '12



Disk (Dirichlet b.c.)
Whitespace in the middle



Hyperbolic surface
No whitespace

A microlocal statement

We assume that (M, g) has Anosov geodesic flow $\varphi_t : S^*M \to S^*M$

$$T(S^*M) = E_0 \oplus E_s \oplus E_u; \quad |d\varphi_t(\rho)v| \le Ce^{-\theta|t|}|v|, \begin{cases} t \ge 0, & v \in E_s(\rho) \\ t \le 0, & v \in E_u(\rho) \end{cases}$$

Using a quantization procedure

$$a \in C_c^\infty(T^*M) \quad \mapsto \quad \operatorname{Op}_h(a) = a(x, \tfrac{h}{i}\partial_x) : L^2(M) \to L^2(M)$$

$$(-\Delta_g - \lambda^2)u = 0 \quad \Longrightarrow \quad (-h^2\Delta_g - 1)u = 0, \quad h := \lambda^{-1}$$

Theorem 1′

Assume that $a|_{S^*M} \ne 0$. Then $\exists C = C(a)$: for all $h \ll 1$, $u \in L^2(M)$

$$\|u\|_{L^2} \le C\|\operatorname{Op}_h(a)u\|_{L^2} + \frac{C\log(1/h)}{h}\|(-h^2\Delta_g - 1)u\|_{L^2}$$

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Remarks

- Implies Theorem 1: $a = a(x) \quad\Longrightarrow\quad \operatorname{Op}_h(a)u = au$
- Sharp: $a|_{S^*M} \equiv 0,\ (-h^2\Delta_g - 1)u = 0 \quad\Longrightarrow\quad \|\operatorname{Op}_h(a)u\| \leq Ch\|u\|$
- Cannot work for $\mathcal{O}(h/\log(1/h))$ quasimodes: Brooks '15, Eswarathasan–Nonnenmacher '17, Eswarathasan–Silberman '17

Applications

- Jin '17: control/observability for Schrödinger equation
- Jin '17, D–Jin–Nonnenmacher '19: exponential energy decay for damped wave equation
- Datchev–Jin WIP, using Jin–Zhang '17: a formula for $C(a)$

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Semiclassical measures

Take a high frequency sequence of Laplacian eigenfunctions

$$(-h_j^2\Delta_g - 1)u_j = 0, \quad \|u_j\| = 1, \quad h_j \to 0$$

We say u_j converges weakly to a measure μ on T^*M if

$$\forall a \in C_c^\infty(T^*M): \quad \langle \mathrm{Op}_{h_j}(a)u_j, u_j\rangle_{L^2} \to \int_{T^*M} a\, d\mu \quad \text{as } j \to \infty$$

Call such limits μ semiclassical measures

Basic properties

- μ is a probability measure, $\operatorname{supp}\mu \subset S^*M$
- μ is invariant under the geodesic flow $\varphi_t : S^*M \to S^*M$
- Natural candidate: Liouville measure $\mu_L \sim d\,\mathrm{vol}$ (equidistribution)
- Natural enemy: delta measure δ_γ on a closed geodesic (scarring)

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Semiclassical measures and Theorem 1

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Theorem 1′: $\quad a|_{S^*M} \not\equiv 0 \quad \Longrightarrow \quad \|\operatorname{Op}_{h_j}(a)u_j\| \geq c > 0$

Theorem 1″

Let μ be a semiclassical measure on M. Then $\operatorname{supp}\mu = S^*M$

Brief overview of history

- Quantum Ergodicity [Shnirelman '74, Zelditch '87, Colin de Verdière '85, Z–Zworski '96]: $\mu = \mu_L$ for density 1 sequence of u_j's
- Quantum Unique Ergodicity conjecture [Rudnick–Sarnak '94]: $\mu = \mu_L$ for all eigenfunctions, that is μ_L is the only semiclassical measure. Proved in the arithmetic case [Lindenstrauss '06]

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Brief overview of history, continued

- Entropy bounds [Anantharaman '08, A–Nonnenmacher '07, Rivière '10, Anantharaman–Silberman '13]: $H_{\mathrm{KS}}(\mu) \geq c_{(M,g)} > 0$, in particular $\mu \neq \delta_\gamma$
- Theorem $1''$: between QE and QUE and 'orthogonal' to entropy bound. There exist μ with $\operatorname{supp}\mu \neq S^*M$, $H_{\mathrm{KS}}(\mu) > c_{(M,g)}$

Main tool: fractal uncertainty principle (FUP)

No function can be localized in both position and frequency near a fractal set

Definition

Fix $\nu > 0$. A set $X \subset \mathbb{R}$ is ν-porous up to scale h if for each interval $I \subset R$ of length $h \leq |I| \leq 1$, there is an interval $J \subset I$, $|J| = \nu|I|$, $J \cap X = \emptyset$

Example: mid-third Cantor set $\mathcal{C} \subset [0,1]$ is $\frac{1}{6}$-porous on scales 0 to 1

Theorem 2 [Bourgain–D '18]

Assume that $X, Y \subset \mathbb{R}$ are ν-porous up to scale h. Then $\exists \beta = \beta(\nu) > 0$:

$$\|\mathbb{1}_X(x)\mathbb{1}_Y(\tfrac{h}{i}\partial_x)\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} = \mathcal{O}(h^\beta) \quad \text{as } h \to 0$$

Note: enough that X, Y be porous up to scales h^{α_X}, h^{α_Y}, $\alpha_X + \alpha_Y > 1$

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- To get rid of the $\log(1/h)$ term need to revise the argument in a way inspired by Anantharaman '08
- We present the proof for the variable curvature case but assume for simplicity (M, g) is hyperbolic, i.e. has curvature -1
- WLOG $a \equiv 1$ on a nonempty open set $\mathcal{U} \subset S^*M$ called the hole

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- Write $I = A_1 + A_\star$, $A_1 = \operatorname{Op}_h(a)$, $\operatorname{WF}_h(A_\star) \cap \mathcal{U} = \emptyset$
- Wave propagator $U(t) = e^{-it\sqrt{-\Delta_g}}$, $U(t)u = e^{-it/h}u$
- $A(t) := U(-t)AU(t) \quad \Longrightarrow \quad \|A_1(t)u\| = \|A_1 u\|$

  $\Longrightarrow \quad u = A_\star(t)u + \mathcal{O}(\|\operatorname{Op}_h(a)u\|)$
- Take $N := \tau\log(1/h)$, $\tau < 1$, use the above for $t = N, \dots, -N$:

$$A^- := A_\star(N)\cdots A_\star(1)A_\star(0), \quad A^+ := A_\star(0)A_\star(-1)\cdots A_\star(-N);$$
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$$\|A^-A^+\|_{L^2\to L^2} = \mathcal{O}(h^\beta), \quad \beta = \beta(\mathcal{U}) > 0$$

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Theorem 1′-weak

Assume that $a \equiv 1$ on a nonempty open $\mathcal{U} \subset S^*M$. Then for $h \ll 1$

$$(-h^2\Delta_g - 1)u = 0 \quad \Longrightarrow \quad \|u\| \leq C\log(1/h)\|\operatorname{Op}_h(a)u\|$$

- Write $I = A_1 + A_\star$, $A_1 = \operatorname{Op}_h(a)$, $\operatorname{WF}_h(A_\star) \cap \mathcal{U} = \emptyset$
- Wave propagator $U(t) = e^{-it\sqrt{-\Delta_g}}$, $U(t)u = e^{-it/h}u$
- $A(t) := U(-t)AU(t) \quad \Longrightarrow \quad \|A_1(t)u\| = \|A_1u\|$

  $\Longrightarrow \quad u = A_\star(t)u + \mathcal{O}(\|\operatorname{Op}_h(a)u\|)$
- Take $N := \tau\log(1/h)$, $\tau < 1$, use the above for $t = N, \dots, -N$:

$$A^- := A_\star(N)\cdots A_\star(1)A_\star(0), \quad A^+ := A_\star(0)A_\star(-1)\cdots A_\star(-N);$$

$$\|u\| \leq \|A^-A^+u\| + C\log(1/h)\|\operatorname{Op}_h(a)u\|$$

- Theorem 1′-weak now follows from the key estimate

$$\|A^-A^+\|_{L^2\to L^2} = \mathcal{O}(h^\beta), \quad \beta = \beta(\mathcal{U}) > 0$$

- $\mathrm{WF}_h(A_\star) \cap \mathcal{U} = \emptyset$ where $\mathcal{U} \subset S^*M$ open nonempty, called the hole
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- Egorov's Theorem $\Longrightarrow$ $A^\pm$ microlocalized in (φ_t = geodesic flow)

$$\Gamma^\pm(N) := \{\rho \in T^*M \mid \varphi_{\mp j}(\rho) \notin \mathcal{U} \quad \text{for all} \quad j = 0, 1, \dots, N\}$$

$\Gamma_-(N)$, $N = 0$ Hole (in white) $\Gamma_+(N)$, $N = 0$

(using Arnold cat map model for the figures)

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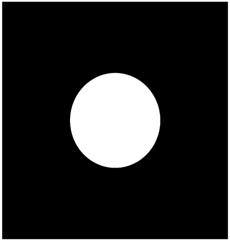

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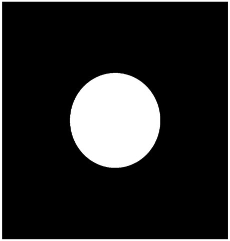

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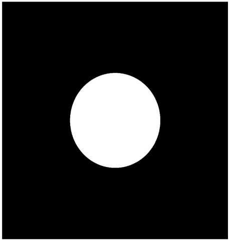

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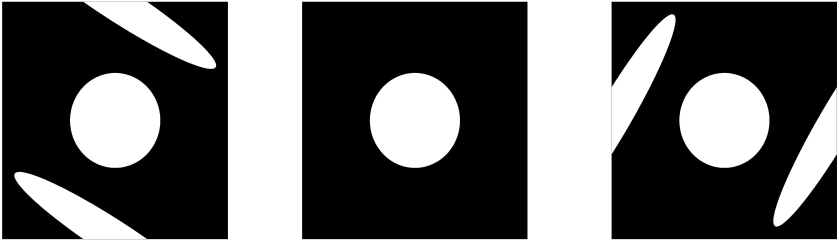

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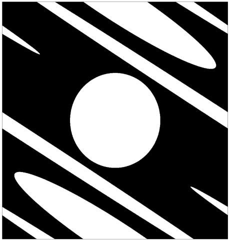
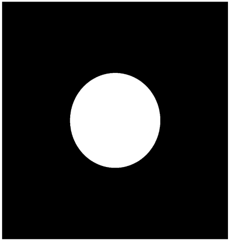


$\Gamma_-(N)$, $N = 2$



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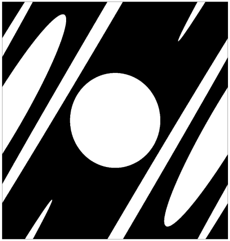

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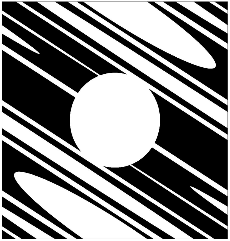
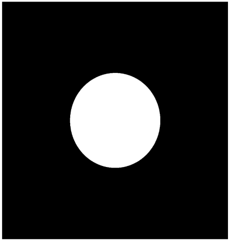


$\Gamma_-(N)$, $N = 3$



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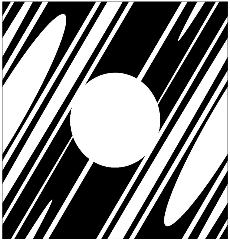

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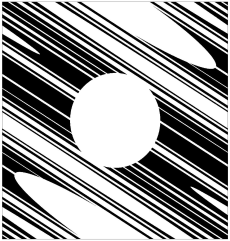
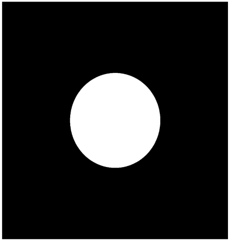


$\Gamma_-(N)$, $N = 4$



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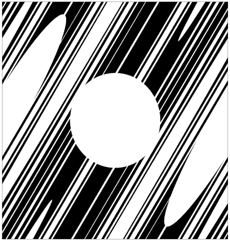

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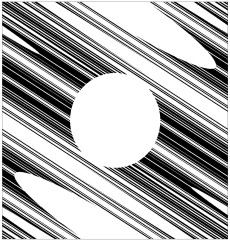
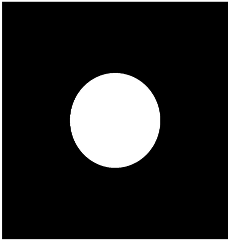


$\Gamma_-(N)$, $N = 5$



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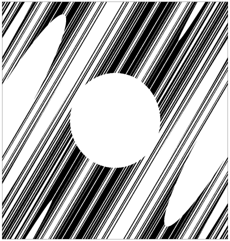
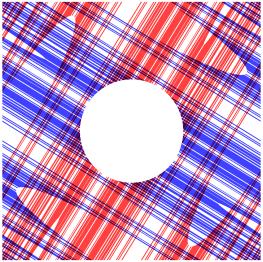


$\Gamma_+(N)$, $N = 5$

(using Arnold cat map model for the figures)

- Key estimate: $\|A^- A^+\|_{L^2\to L^2} = \mathcal{O}(h^\beta)$, $A^\pm$ microlocalized on $\Gamma^\pm(N)$, $N = \tau \log(1/h)$
- Γ^+ smooth in the unstable direction, porous up to scale h^τ in the stable direction
- Same true for Γ^-, switching stable/unstable
- The product $A^- A^+$ is not pseudodifferential
- Will use FUP to show the key estimate



Challenges in variable curvature

- Variable expansion rates of the flow φ_t $\implies$ take a dynamically fine partition $A_\star = A_2 + \dots + A_L$ and put $N =$ local Ehrenfest time for each word
- Stable/unstable foliations are not C^∞ $\implies$ cannot make $A^\pm$ pseudodifferential following D–Zahl '16

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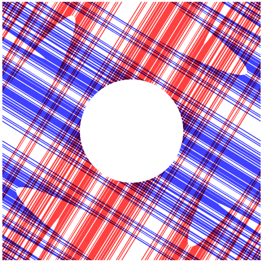

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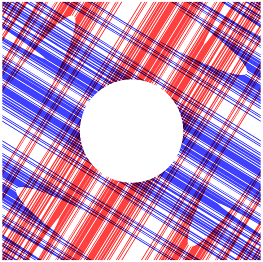

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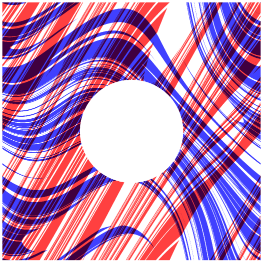
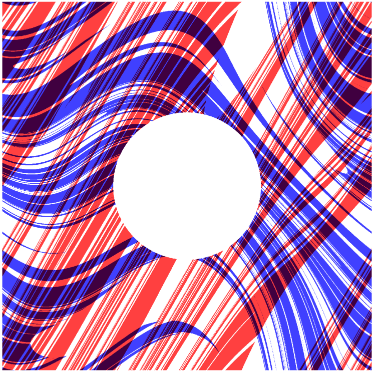
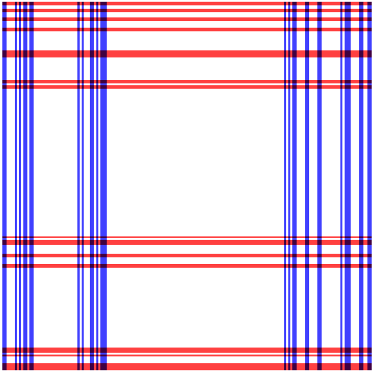


Reduction to FUP



$\overset{?}{\Longleftarrow}$



$\|A^- A^+\|_{L^2(M)\to L^2(M)} = \mathcal{O}(h^\beta)$ $\qquad$ $\|\mathbb{1}_X(x)\mathbb{1}_Y(\frac{h}{i}\partial_x)\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} = \mathcal{O}(h^\beta)$

- Restrict to S^*M, remove the flow direction: 2D $\Longleftarrow$ 1D
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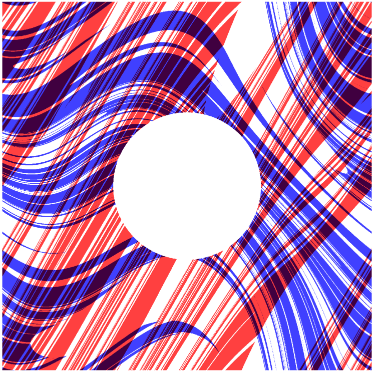

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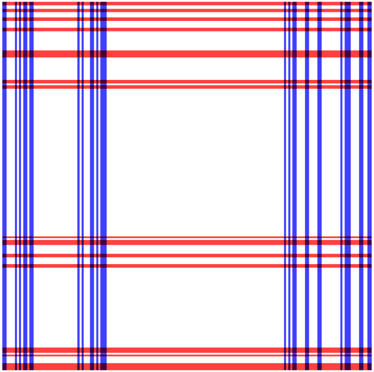

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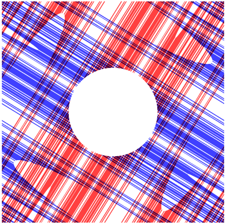
Cluster decomposition

- Replace A^- by $\widetilde{A}^-$ which microlocalizes to an $h^{1/6}$ neighborhood of Γ^-
- Write $A^+ = \sum_j A_j^+$ where each A_j^+ microlocalizes to an $h^{2/3}$ neighborhood of some unstable leaf
- $h^{1/6} \cdot h^{2/3} \gg h \quad \Longrightarrow \quad B_j := \widetilde{A}^- A_j^+$ are almost orthogonal:

$$\|B_j^* B_{j'}\|_{L^2 \to L^2}, \|B_{j'} B_j^*\|_{L^2 \to L^2} = \mathcal{O}(h^\infty) \quad \text{when } |j - j'| \gg 1$$

- By Cotlar–Stein enough to show

$$\max_j \|\widetilde{A}^- A_j^+\|_{L^2 \to L^2} = \mathcal{O}(h^\beta)$$



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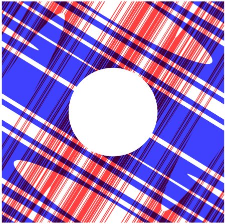

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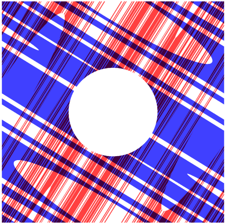

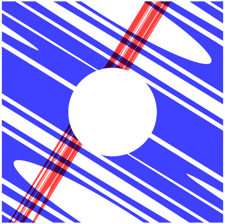

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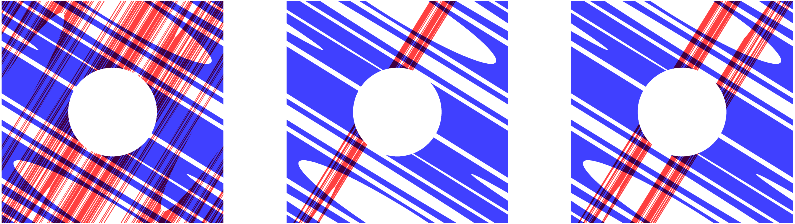

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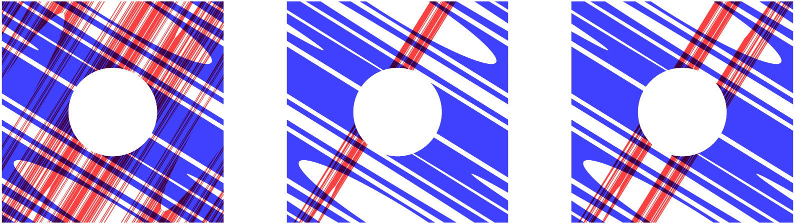

- Need $\|\widetilde{A}^- A_j^+\|_{L^2\to L^2} = \mathcal{O}(h^\beta)$; $\widetilde{A}^- \leftrightarrow \widetilde{\Gamma}^- := h^{1/6}$ neighborhood of Γ^-, $A_j^+ \leftrightarrow \Gamma_j^+ := \Gamma^+ \cap (h^{2/3}$-neighborhood of some unstable leaf $W_j)$
- As before, restrict to S^*M and remove the flow direction
- Unstable foliation has $C^{2-} \subset C^{3/2}$ regularity [Hurder–Katok '90] $\implies$ construct C^∞ symplectomorphism $\varkappa$ to $T^*\mathbb{R}$ s.t. unstable leaves $h^{2/3}$-close to W_j are mapped h-close to horizontal lines
- Then $\varkappa(\Gamma_j^+) \subset \{\xi \in \Omega^+\}$, $\varkappa(\widetilde{\Gamma}^- \cap \Gamma_j^+) \subset \{x \in \Omega^-\}$ where $\Omega^+, \Omega^- \subset \mathbb{R}$ are porous on scales up to h, $h^{1/6}$
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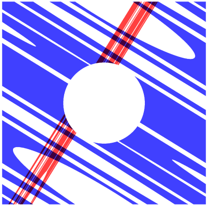

$\xrightarrow{\varkappa}$

- Need $\|\widetilde{A}^- A_j^+\|_{L^2\to L^2} = \mathcal{O}(h^\beta)$; $\widetilde{A}^- \leftrightarrow \widetilde{\Gamma}^- := h^{1/6}$ neighborhood of Γ^-, $A_j^+ \leftrightarrow \Gamma_j^+ := \Gamma^+ \cap (h^{2/3}$-neighborhood of some unstable leaf $W_j)$
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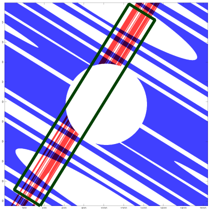

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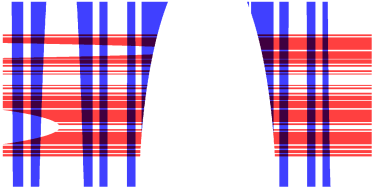

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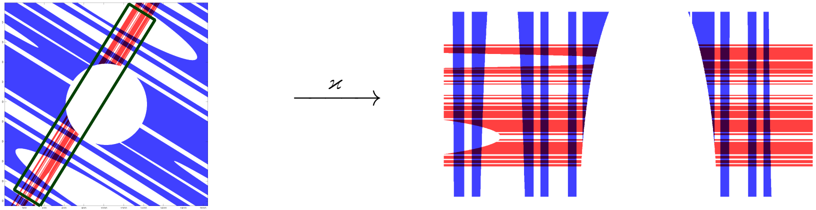

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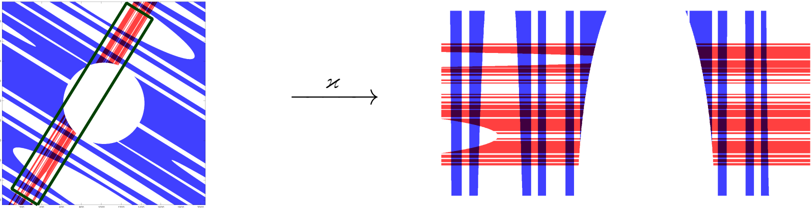

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- To make the above arguments rigorous, use Egorov's Theorem up to local Ehrenfest time (adapted from Rivière '10) and long logarithmic time propagation of Lagrangian states due to Anantharaman '08, Anantharaman–Nonnenmacher '07, Nonnenmacher–Zworski '09

Thank you for your attention!

Happy Birthday Alexander!