"ERGODIC PROPERTIES OF EIGENFUNCTIONS AFTER SHNIRELMAN"

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SATURDAY FEB 27 2021

ALEXANDER SHNIRELMAN'S

75TH BIRTHDAY.

SOME PAPERS OF SASHA SHNIRELMAN

1) "ERGODIC PROPERTIES OF EIGENFUNCTIONS" USPEHI MATH NAUK 29, 6, 181-182 (1974).

2) "STATISTICAL PROPERTIES OF EIGENFUNCTIONS" PROC ALL USSR SCHOOL OF DIFFERENTIAL EQUATIONS, ARMENIA 1978 (8 - PAGES).

3). "ON THE ASYMPTOTIC MULTIPLICITY OF THE SPECTRUM OF THE LAPLACE OPERATOR" USPEHI MATH NAUK 30, 4, 265-266 (1975)

4) EXPOSITIONS WITH PROOFS OF (1) AND (3) IN TWO APPENDICES IN THE BOOK BY LAZUTKIN "KAM THEORY AND SEMI CLASSICAL APPROXIMATION TO EIGENFUNCTIONS".

THE SETTING OF THESE PAPERS IS THE HAMILTONIAN WHICH IS THE GEODESIC MOTION g_t ON THE UNIT (CO)TANGENT BUNDLE $T_1^*(X)$ OF A COMPACT RIEMANNIAN MANIFOLD X.

THE SPECTRUM IS THAT OF THE LAPLACIAN ON $L^2(X)$.

$$\Delta \phi_j + t_j^2 \phi_j = 0$$

$$0 = t_1 < t_2 \le t_3 \cdots \qquad \phi_j \text{ O.N.B OF EIGEN-FUNCTIONS}.$$

WE RESTRICT FURTHER TO $\dim X = 2$.

DURING THE PERIOD (1970's) THE CONSTRUCTION OF APPROXIMATE EIGENFUNCTIONS "QUASI-MODES" ON X ASSOCIATED WITH STABLE PERIODIC ORBITS OF g_t OR ASSOCIATED WITH INVARIANT TORI IN THE KAM SETTING WAS VERY ACTIVE (LAZUTKIN, RALSTON, COLIN-DE-VERDIERE, ...)

PAPER (3) OF SASHA IS ALONG THESE LINES, SHOWING THAT FOR ANY METRIC CLOSE TO THE FLAT ONE ON THE 2-TORUS AND ANY N, THERE IS C_N SUCH THAT; $\min(t_k - t_{k-1}, t_{k+1} - t_k) \le C_N t_k^{-N}$.

SO THAT THE "DESIRE" THAT A GENERIC METRIC ON THE TORUS (WHOSE SPECTRUM IS SIMPLE-UHLENBECK) SATISFY A DIOPHANTINE SPACING CONDITION FAILS.

THE FIRST (TWO PAGE) PAPER CAME OUT OF THE BLUE AND HAS THAT ORIGINALITY CHARACTERISTIC OF SASHA'S WORK.

CORRESPONDING TO THE EIGENFUNCTIONS DEFINE THE PROBABILITY MEASURES

$$\mu_t = |\phi_t(x)|^2 dA(x) \quad \text{ON } X \qquad \text{---} (1)$$

AND ITS SHNIRELMAN MICRO-LOCAL LIFT

$$\nu_t \quad \text{ON} \quad T_1^*(X).$$

FOR $a \in C^\infty(T_1^*(X))$ WHICH WE VIEW AS A DEGREE ZERO FUNCTION ON $T^*(X)$ LET $Op(a)$ BE A CORRESPONDING ψ.D.O. WITH SYMBOL a.

$$\nu_t(a) = \langle Op(a) \phi_t, \phi_t \rangle \qquad \text{---} (2)$$

ASYMPLOTICALLY AS $t \to \infty$ THIS WILL NOT DEPEND ON THE CHOICE OF $Op(a)$, AND BY A SYMMETRIZATION (FRIEDRICHS) ONE CAN TAKE ν_t TO BE A POSITIVE MEASURE.

SHNIRELMAN'S QUANTUM ERGODICITY THEOREM

IF THE GEODESIC FLOW ON X IS ERGODIC W.R.T THE VOLUME FORM (LIOUVILLE) μ ON $T_1^*(X)$, THEN FOR ALMOST ALL t_j's, $j=1,2,\ldots$ IN THE SENSE OF FULL DENSITY; $\nu_{t_j} \longrightarrow \mu$.

SOME HISTORY:

- THE 1974 ANNOUNCEMENT HAS LITTLE IN THE WAY OF PROOFS.
- THE 1973 PAPER WAS KNOWN TO FEW PEOPLE (AT LEAST FOR SOME YEARS).
- COLIN-DE-VERDIERE GAVE A REPORT ON SHNIRELMAN'S THEOREM IN THE ECOLE POLYTECHNIQUE P.D.E SEMINAR <u>1984-85</u>, IN WHICH HE GIVES A COMPLETE AND ELEGANT PROOF AS WELL AS SOME CLARIFICATIONS OF THIS "REMARKABLE THEOREM".
- ZELDITCH AROUND THE SAME CONSTRUCTED A DETAILED PROOF IN THE CASE THAT $X = \Gamma \backslash \mathbb{H}$ (A COMPACT HYPERBOLIC SURFACE) SO $T_1^*(X) = \Gamma \backslash SL_2(\mathbb{R})$, AND HE DEVELOPS A CANONICAL QUANTIZATION ~~USING~~ AND Ψ.D. CALCULUS USING SOME REPRESENTATION THEORY OF $SL(2,\mathbb{R})$.

GENERALIZATIONS TO X WITH BOUNDARY WERE DEVELOPED BY GERARD-LEIGHTMAN, SEMI CLASSICAL VERSIONS BY HELFFER-ROBERT-MARTINEZ AND ZELDITCH-ZWORSKI, ...

SOME INGREDIENTS:

(1) FOR $\tau \in \mathbb{R}$, $\langle e^{-i\sqrt{\Delta}\tau} Op(a) e^{i\sqrt{\Delta}\tau} \phi_t, \phi_t \rangle$ $= \nu_t(a)$, AND BY EGOROV THE LHS IS APPROXIMATED BY $\langle Op(g_\tau a)\phi_t, \phi_t \rangle$ AS $t \to \infty$.

HENCE ANY WEAK* LIMIT OF THE ν_j's "A QUANTUM LIMIT" MUST BE g_τ INVARIANT!

(2) THE ERGODICITY IS USED IN SOME FORM AS

$$\frac{1}{T}\int_0^T f(g_t v)\,dt \to \int_{T_1^*(X)} f(v)\,d\mu(v), \text{ a.e.}$$

(3) A TAUBERIAN ARGUMENT IS USED TO GET FULL DENSITY.

ZELDITCH EXECUTES STEP (3) WITH A VARIANCE ARGUMENT: FOR $a \in C^\infty(T_1^*(X))$ ~~DEFINE~~ THE "QUANTUM VARIANCE"

$$V(a,T) := \sum_{t_j \le T} |\nu_{t_j}(a) - \mu(a)|^2 = o\Big(\sum_{t_j \le T} 1\Big) \text{ AS } T \to \infty$$

$$N(T) = \sum_{t_j \leq T} 1 \sim \frac{\text{AREA}(X)}{4\pi} T^2 \quad (\text{WEYL}).$$

IMMEDIATE QUESTIONS:

(1) WHAT ARE THE POSSIBLE QUANTUM LIMITS? THEY MUST BE g_t INVARIANT.

(2) NUMERICAL EXPERIMENTS FOR THE BUNIMOVICH (ERGODIC) STADIUM (BY THE 1980' THERE WERE MANY NUMERICAL EXPERIMENTS, AND CURIOUSLY SASHA REFERENCES THIS EXAMPLE IN HIS TWO PAGES!) SHOW THAT THERE ARE CLEARLY BOUNCING BALL QUANTUM LIMITS,

BUT ALSO A NEW SUGGESTION OF "SCARRING" ON UNSTABLE PERIODIC ORBITS.

A. HASSEL (2008) SHOWED THAT THE STADIUM HAS A NON LIOUVILLE QUANTUM LIMIT.

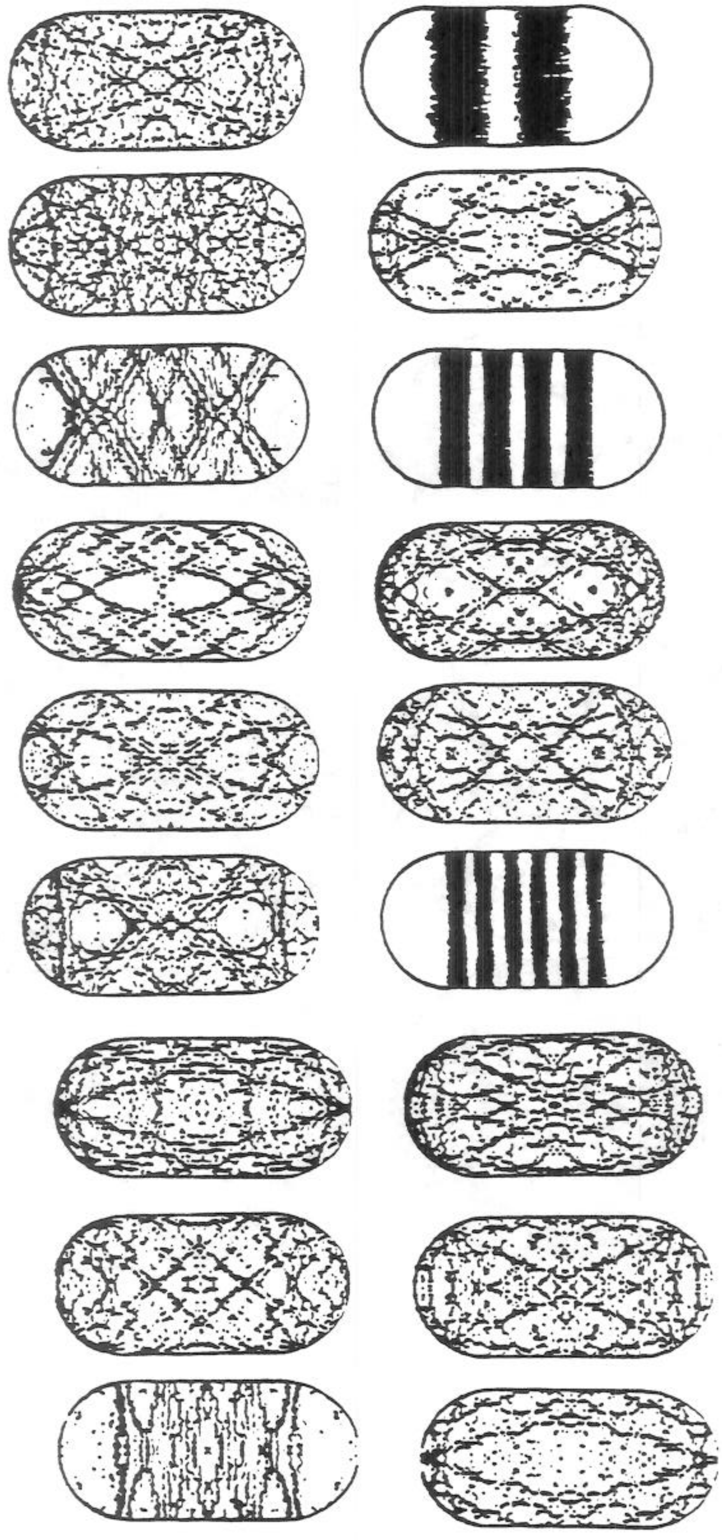


Figure 2.3. Density plot of $|\Phi(x)|^2$ for eigenstates of the stadium (Black signifies high density) for eigenvalues $\sqrt{\lambda} = k$, where going from top to bottom, $k = 110.389$, 119.413, 119.417, 119.451, 119.499, 119.512, 119.512, 119.525, 119.547, 119.587, 119.637, 119.672, 119.691, 119.701, 119.740, 119.802, 119.809, 119.839.

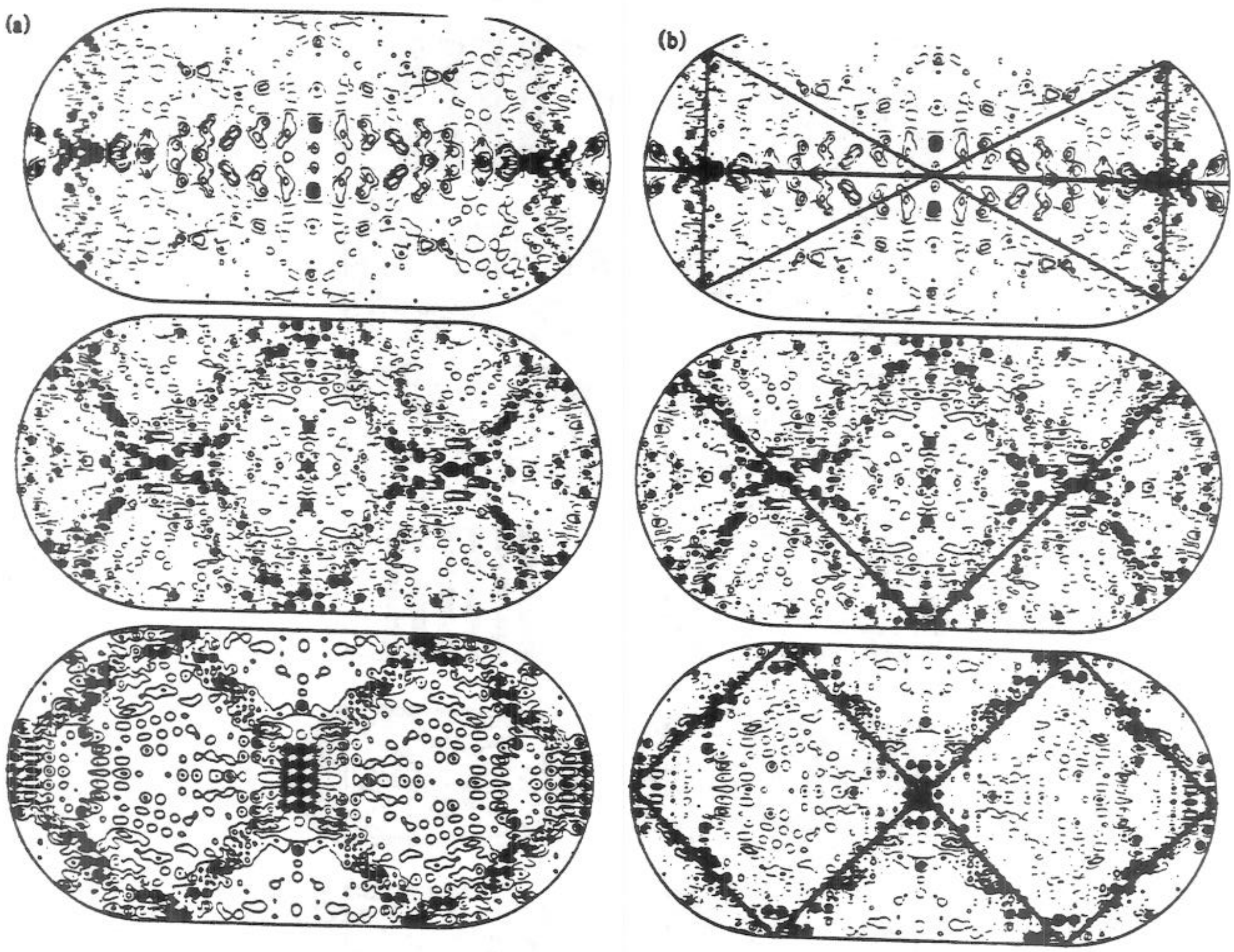
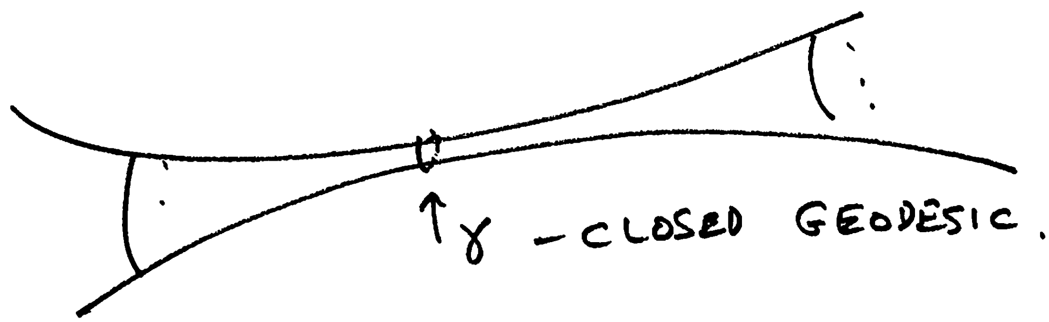


Figure 2.4. Left column, three scarred states of the stadium; right column, the isolated, unstable periodic orbits corresponding to the scars.

TO AVOID SUCH THINGS AS BOUNCING BALLS WE RESTRICT TO UNIFORMLY HYPERBOLIC DYNAMICS THAT IS X HAS NEGATIVE CURVATURE. G_τ IS ERGODIC AND THE ONLY CLOSED ORBITS ARE COUNTABLY MANY CLOSED GEODESICS.

COLIN DE VERDIERE (1987) CRYSTALIZES THE QUESTION AS TO WHETHER THE MOST SINGULAR g_τ INVARIANT MEASURE, THE ARC LENGTH ON A PERIODIC GEODESIC CAN BE A QUANTUM LIMIT?



HE SHOWS THAT ONE CANNOT BUILD A QUASI-MODE LOCALLY ABOUT SUCH AN UNSTABLE γ, BUT THAT μ_γ CAN OCCUR AS A QUANTUM LIMIT (NOT ~~with~~ IN A CASE WHERE $K<0$).

FOR X OF NEGATIVE CURVATURE ZELDITCH SHOWS THAT THE QUANTUM VARIANCE DECAYS WITH A RATE:

$$V(a,T) \ll_T \frac{N(T)}{\log T} \quad \text{AS } T\to\infty.$$

IN ORDER TO ADDRESS THESE INDIVIDUAL QUESTIONS FOR $K<0$, ONE NEEDS TO BREAK CERTAIN SEMI-CLASSICAL THRESHOLDS.

ONE BASIC ISSUE IS MULTIPLICITIES OF THE t'S, WHILE WE (OR I SHOULD SAY I) BELIEVE THAT THESE ARE SMALL $O_\varepsilon(t^\varepsilon)$, NOTHING BETTER THAN $O(t/\log t)$ IS KNOWN (IN DIMENSION 2).

THE DEVELOPMENT HAVE SPLIT IN TWO DIRECTIONS:

(1) ARITHMETIC: SPECIALIZE TO $X = \Gamma\backslash\mathbb{H}$ AN ARITHMETIC HYPERBOLIC SURFACE.

- THESE ARE (ESSENTIALLY) CHARACTERIZED BY HAVING CORRESPONDENCES (MULTI VALUED SYMMETRIES) AND WITH THESE A NATURAL FAMILY OF "HECKE" OPERATORS. WHICH COMMUTE WITH Δ AND WITH EACH OTHER.
- SO FOR THESE THERE IS A CANONICAL O.N. BASIS OF EIGENFUNCTIONS WHICH WE USE - GETS AROUND MULTIPLICITY
- IN THESE CASES WE CAN BRING NUMBER THEORY INTO THE STUDY - AND WE DEMAND STRONGER RESULTS.

(2) WHAT CAN BE SAID IN GENERAL ?

ARITHMETIC CASE :

(1) RUDNICK-S (1993) EXAMINED THE COLIN DE VERDIERE CLOSED GEODESIC QUESTION AND SHOWED THAT FOR ANY QUANTUM LIMIT ν ON AN ARITHMETIC SURFACE SINGSUPPORT ν CANNOT BE CONTAINED IN A UNION OF CLOSED GEODESICS.

BASED ON THIS AND VARIOUS OTHER DEVELOPMENTS BELOW, WE CONJECTURED

QUE QUANTUM UNIQUE ERGODICITY:

THE ONLY QUANTUM LIMIT POSSIBLE IN THE $K<0$ SETTING IS μ.

THERE IS A FUNDAMENTAL IDENTITY FOR THE SHNIRELMAN MICRO LOCAL LIFTS IN THE ARITHMETIC CASE THAT ALLOWS FOR A TRANSLATION OF THE BASIC PROBLEM TO ONE OF L-FUNCTIONS.

IF a ITSELF IS A HECKE EIGENFORM
ON $\Gamma\backslash SL(2,\mathbb{R}) = \Gamma\backslash T_1^*(X)$ THEN

<u>T. WATSON'S FORMULA</u> (PRINCETON THESIS 1994):

$$|\nu_{t_j}(a)|^2 = (*) \; L\left(\tfrac{1}{2}, \pi_j \times \pi_j \times \pi_a\right)$$

EXPLICIT

THE L-FUNCTION OF THE TRIPLE PRODUCT OF
THE FORMS CORRESPONDING TO ϕ_{t_j} AND a, AT THE
CENTER OF ITS CRITICAL STRIP.

PROGRESS? YES IF WE BELIEVE
THE (GRAND) RIEMANN HYPOTHESIS.

GRH $\Rightarrow$ IF $\int a = 0$,

$$|\nu_{t_\phi}(a)| \ll_{\varepsilon, a} t_\phi^{-1/2+\varepsilon}.$$

AND THIS IS AN OPTIMAL DECAY RATE!

FOR QUE FOR X ONE <u>ONLY</u> NEEDS
A SUBCONVEX BOUND FOR THE L-FUNCTION.

FOR SPECIAL FORMS ϕ_t THIS WAS ESTABLISHED

- BY LUO-S, JAKOBSON FOR THE CONTINUOUS SPECTRUM (WHEN X IS NOT COMPACT) 90's
- FOR DIHEDRAL FORMS ϕ_t (THESE COMPRISE ABOUT ~~FE~~ T OF THE T^2 FORMS ~~&~~ $t \leq T$) S, 90'.

BUT THE GENERAL SUBCONVEXITY HAS RESISTED ALL EFFORTS SO FAR.

MEASURE RIGIDITY:

A QUANTUM LIMIT ν ON ~~X~~ $\Gamma \backslash SL_2(\mathbb{R})$ IS $g_t : \Gamma g \to \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$ INVARIANT. COULD IT HAVE FURTHER INVARIANCE PROPERTIES THANKS TO BEING A HECKE EIGENFUNCTION?

TO UNDERSTAND THIS CONSIDER

$$Y = \Gamma \backslash \mathbb{H} \times \mathbb{H}, \quad \Gamma \leq SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$$

AN IRREDUCIBLE LATTICE.

ON Y WE HAVE Δ_1, Δ_2 THE LAPLACIANS IN z_1 RESP z_2 ($\Delta = \Delta_1 + \Delta_2$), AND WE NATURALLY ASK FOR JOINT EIGENFUNCTIONS

$$\phi_t(z_1, z_2) \quad , \quad t = (t_1, t_2).$$

• ONE CAN FORM A SHNIRELMAN "MICRO-LOCAL" LIFT OF $|\phi_t(z_1,z_2)|^2 dV(z)$ TO

$\Gamma \backslash SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ AND WE HAVE EXTRA INVARIANCE OF A QUANTUM LIMIT, ν NAMELY UNDER

$$\Gamma g \longrightarrow \Gamma g\left(\begin{pmatrix} e^{t_1/2} & 0 \\ 0 & e^{-t_1/2}\end{pmatrix}, \begin{pmatrix} e^{t_2/2} & 0 \\ 0 & e^{-t_2/2}\end{pmatrix}\right).$$

• THAT IS ν IS INVARIANT UNDER A HIGHER RANK DIAGONAL ACTION.

• THANKS TO A REMARKABLE CONJECTURE OF FURSTENBERG, ONE EXPECTS THAT UNLIKE THE GEODESIC FLOW CASE, ONCE THE RANK IS TWO OR MORE, THINGS RIGIDIFY AND THE ONLY INVARIANT MEASURES SHOULD BE THE OBVIOUS ONES!

THIS CONJECTURE IN THE GENERAL ~~SETTING~~ OF $\Gamma\backslash G$ HAS BEEN PROVEN BY EINSIEDLER–KATOK AND LINDENSTRAUSS UNDER THE ADDED ASSUMPTION THAT THE MEASURE ν HAS POSITIVE ENTROPY FOR SOME ELEMENT IN THE FLOW.

LINDENSTAUSS DEVELOPS THE THEORY IN THE CONTEXT OF $SL_2(\mathbb{R}) \times SL_2(\mathbb{Q}_p)$ (p-th HECKE OPERATOR) AND COUPLED WITH A PROOF (JOINT WITH BOURGAIN) THAT EVERY ERGODIC COMPONENT OF A QUANTUM LIMIT IN THIS SETTING MUST HAVE POSITIVE ENTROPY $\Longrightarrow$

THEOREM (LINDENSTRAUSS)

THE QUE CONJECTURE IS TRUE FOR ARITHMETIC X's.

HOW ABOUT THE QUANTUM VARIANCE IN THE ARITHMETIC SETTING?

CLASSICAL VARIANCE OF g_t (RATNER)

X, ~~ABEGC~~ $K<0$, $a \in C^\infty(T_1^* X)$, $\int a \, d\mu = 0$

THEN

$$\frac{1}{\sqrt{T}} \int_0^T a(g_\tau v) \, d\tau$$

FOR a.e. v IS GAUSSIAN WITH MEAN 0 AND VARIANCE A QUADRATIC FORM IN a:

$$V_{CLASSICAL}(a) = \int_{-\infty}^{\infty} \int_{T_1^*(X)} a(g_\tau v) a(v) \, d\mu(v) \, dt$$

RECALL THAT GRH $\Rightarrow \langle Op(a)\phi_j, \phi_j \rangle \ll t_j^{-1/2+\varepsilon}$

THEOREM (LUO-S, ZHAO-S, NELSON)

$\int a \, d\mu = 0$,

$$VAR(a,T) = \sum_{t_j \le T} |\langle Op(a)\phi_j, \phi_j \rangle|^2 \sim V_{QUANTUM}(a) \, T$$

AS $T \to \infty$.

MOREOVER THE QUADRATIC FORMS

$V_{CLASSICAL}(a)$ AND $V_{QUANTUM}(a)$ ARE

DIAGONALIZED ON $L^2_0(\Gamma \backslash SL_2(\mathbb{R}))$ BY

THE IRREDUCIBLE REPRESENTATIONS π_λ AND

THEY AGREE UP TO A SCALAR MULTIPLE

NAMELY $L(\frac{1}{2}, \pi)$!

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2) GENERAL CASE OF VARIABLE CURVATURE

FOR A LONG TIME THERE WAS LITTLE PROGRESS IN THE GENERAL CASE BUT THAT CHANGED WITH THE BREAKTHROUGH OF ANANTHARAMAN AND HER FOLLOWUP WORK WITH NONENMACHER WHICH GAVE QUANTITATIVE BOUNDS. BY A DIRECT COMBINATORIAL ESTIMATION OF THE ENTROPIES OF THE Y_t'S, EXTENDING THE RANGE OVER THE EHRENFEST TIME THRESHOLD TO GIVE LOWER BOUNDS FOR QUANTUM LIMITS OF QUASI MODES:

THEOREM (ANANTHARAMAN 2008).

X A COMPACT ~~RE~~ RIEMANNIAN MANIFOLD OF NEGATIVE SECTIONAL CURVATURE AND LET ν BE A QUANTUM LIMIT, THEN THE (KOLMOGOROV-SINAI) ENTROPY $h(\nu)$ IS POSITIVE.

THIS FINALLY RESOLVES COLIN DE VERDIERE'S QUESTION SINCE CLEARLY ν CANNOT BE ~~THE~~ ~~SI~~ A CONVEX AVERAGE OF THE SINGULAR MEASURES ON CLOSED GEODESICS (AS THESE HAVE ZERO ENTROPY).

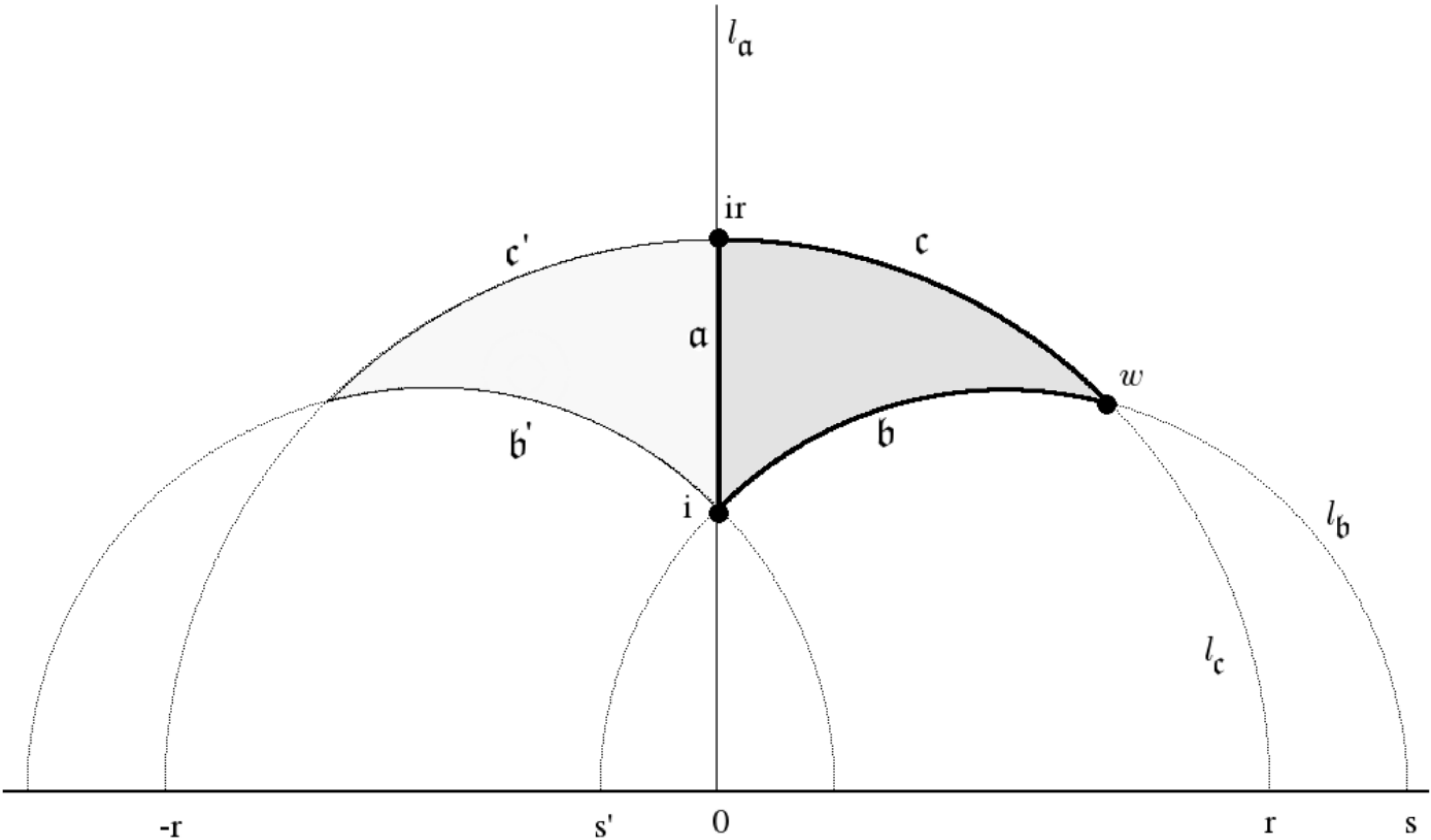
ANOTHER DEVELOPMENT USING A FRACTAL UNCERTAINTY PRINCIPLE ALLOWS ONE TO PROVE IN THIS GENERAL SETTINK ($K<0$) THAT QUANTUM LIMITS MUST HAVE FULL ~~SU~~ TOPOLOGICAL SUPPORT — SEE DYATLOV LECTURE IN 5 MINS.

THERE ARE MANY APPLICATIONS OF QUANTUM ERGODICITY AND QUANTUM UNIQUE ERGODICITY. WE MENTION A COUPLE TO THE PROBLEM OF COUNTING NODAL DOMAINS IN WHICH THE ABOVE ARE USED AS ONE OF THE CRITICAL INGREDIENTS.

LET T BE AN ARITHMETIC TRIANGLE IN THE HYPERBOLIC PLANE

<u>THEOREM</u> (GHOSH-REZNIKOV-S CONDITIONAL ON RH
JANG-JUNG UNCONDITIONAL)
2018

IF ϕ_t IS A DIRICHLET OR NEUMANN EIGENFUNCTION ON T THEN THE NUMBER OF NODAL DOMAINS OF ϕ_t GOES TO INFINITY AS $t \to \infty$.



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SIMILARLY THE QUANTUM ERGODIC RESTRICTION THEOREM OF TOTH, CHRISTIANSON, ZELDITCH AND HASSEL IS ONE OF THE INGREDIENTS IN:

THEOREM (HEZARI 2018)
LET X BE COMPACT RIEMANNIAN SURFACE WITH PIECEWISE SMOOTH BOUNDARY. IF THE BILLIARD FLOW ON X IS ERGODIC, THEN THERE IS A DENSITY ONE SUBSEQUENCE OF ~~BOTH~~ EACH OF THE DIRICHLET AND NEUMANN EIGENFUNCTIONS FOR WHICH THE NUMBER OF NODAL DOMAINS GOES TO INFINITY ALONG THE SEQUENCE.

IT IS DIFFICULT TO IMAGINE SUCH A THEOREM AS THE LAST IF IT WAS NOT FOR SASHA'S TWO PAGE PAPER IN 1974.

HAPPY BIRTHDAY SASHA AND WE WISH YOU MANY MORE!