**Eigenfunctions on the hyperbolic surfaces: from scars to random matrices**

**1.** The association of the spectrum of Laplacian on the surfaces of negative curvature, as well as in the compact domains with concave boundaries, with random matrices is quite mysterious. After all, those matrices should be **made of something**; they should have rows, and columns, and entries, and some kind of randomness -- where is it all?

This talk presents my attempt to produce a random matrix associated with Laplacian on a hyperbolic surface. The main character of this story is a **scar**, i.e. a condensation of an eigenfunction around an unstable periodic trajectory.

**2.** Here is an example of a scar in the bowtie-shaped domain:

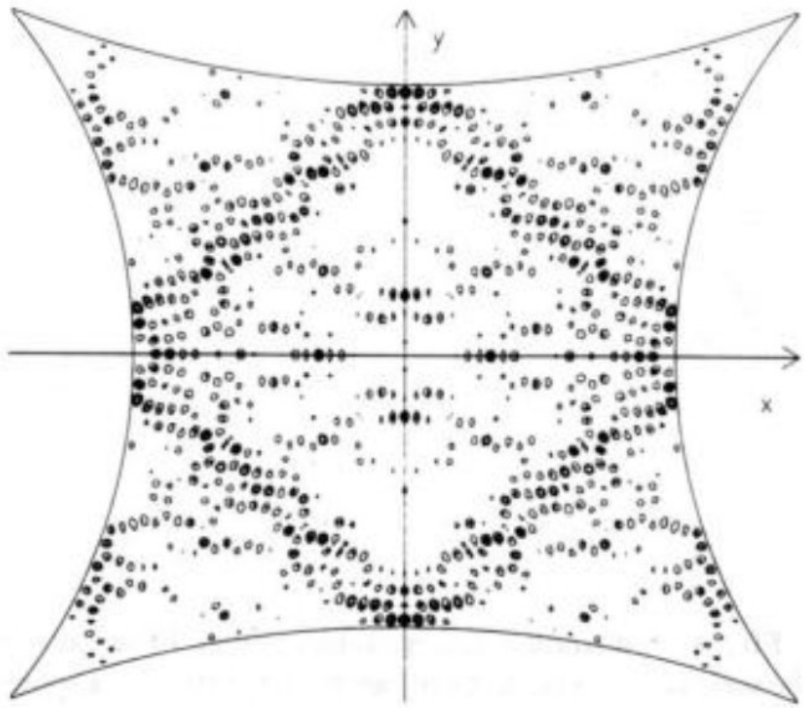


Figure 1. Diamond-shaped scar in a bowtie domain.

**3.** The very existence of the scars is far from evident. For **stable** periodic trajectories Babich and Lazutkin constructed "quasimodes" concentrated in a narrow neighborhood of them, and rapidly decreasing with distance from them:

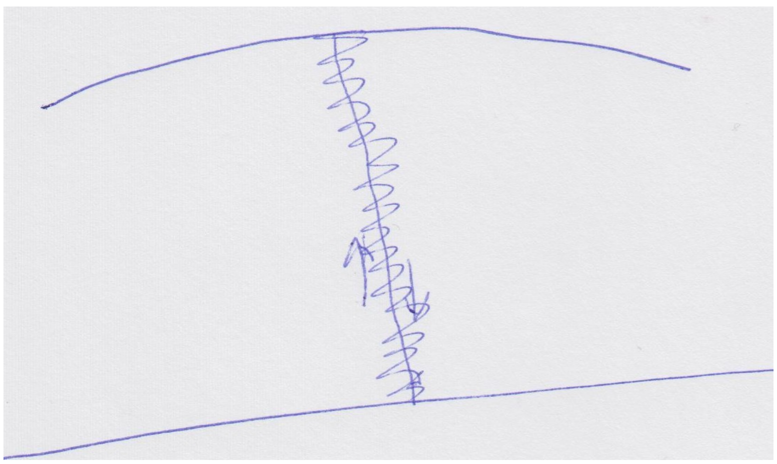

Figure 2. Eigenfunction concentrated along a stable periodic trajectory

But how can the energy of an eigenfunction concentrate near the unstable trajectory?

To see what happens in this case, consider a model case. Consider a hyperbolic surface of revolution:

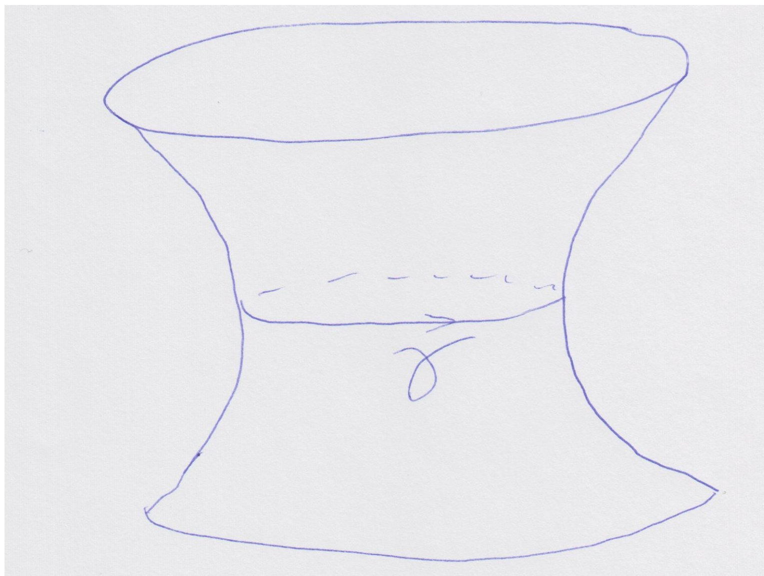


Figure 3. Minimal trajectory on a hyperbolic surface of revolution

Let $\gamma$ be the geodesic of minimal length around the neck. Suppose the length of $\gamma$ is $2\pi$. Let us find the eigenfunctions of Laplacian near $\gamma$. Let $x$ and $y$ be coordinates along and across $\gamma$. The variables separate, and we can look for solution of the equation $\Delta u - \lambda u = 0$ in the form

$$u(x, y) = e^{ikx} g(\eta) .$$

where $\eta=\sqrt{k}\,y$ . The function $g(\eta)$ satisfies approximately the Weber equation

$$\frac{d^2 g}{d\eta^2}+(\eta^2+a)g=0 \ ,$$

where $a$ is such that $\lambda=-k^2+k\,a$ . There are two linearly independent solutions, $w_0(\eta,a)$ (even), and $w_1(\eta,a)$ (odd).

There is a principal difference between the eigenfunctions concentrated around stable and unstable trajectories. For stable trajectories the equation has the form

$$\frac{d^2 g}{d\eta^2}-(\eta^2-a)g=0 \ ,$$

and bounded solutions exist only if $a=m+1/2$ , $m\geq 0$ . Hence, the eigenvalues are discrete and have the form $\lambda=-k^2-(m+1/2)k$ .

In contrast to this, the Weber functions are bounded for any $a$ , and for any $\lambda$ there exists a countable set of the Weber eigenfunctions around $\gamma$ , labeled by $k$ .

To have an idea how these eigenfunctions look, I've plotted them for $a=0$ , $a=3$ , and $a=-3$ .Here they are:

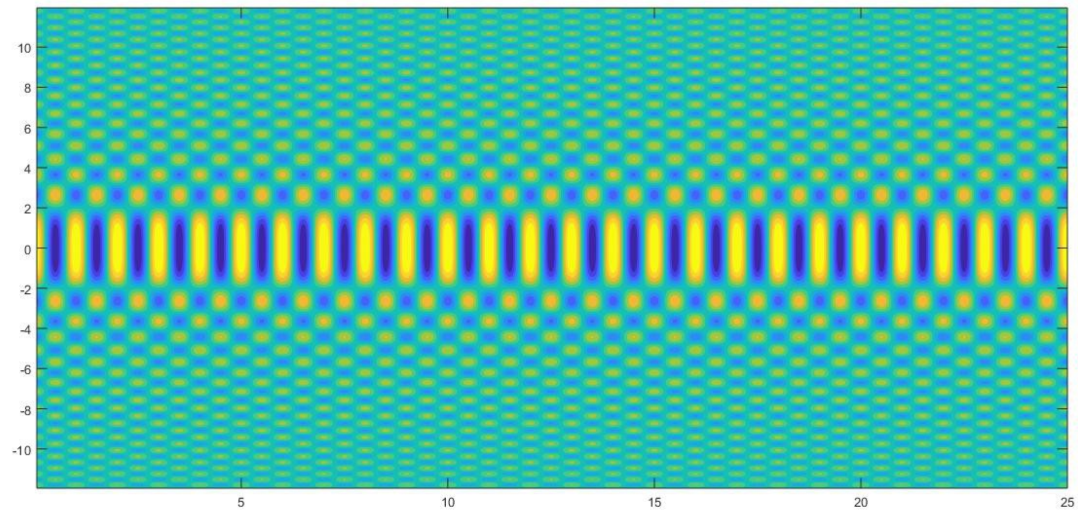


Figure 4. Scar function for $a=0$ (a perfect scar).

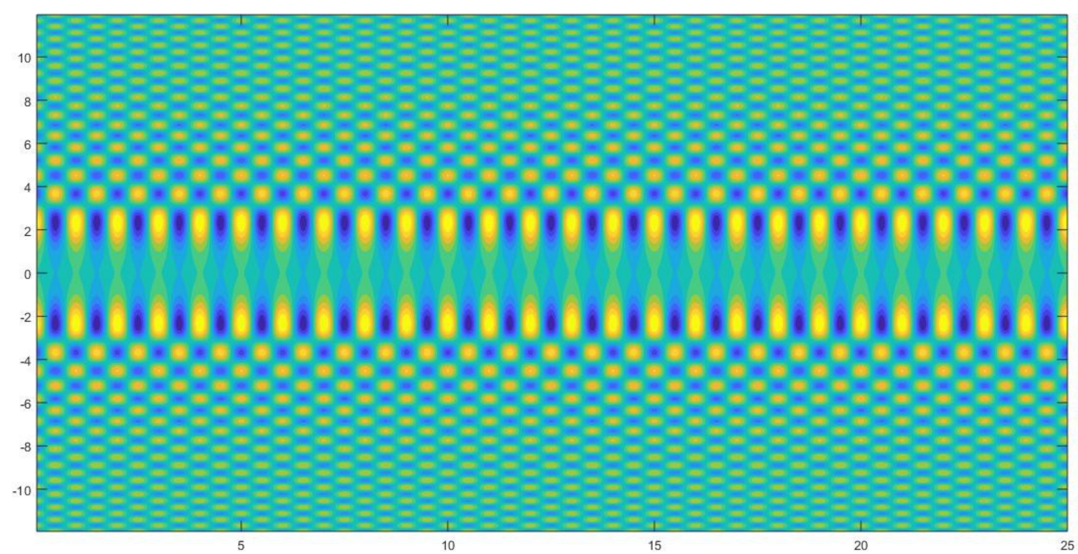


Figure 5. Scar function for $a=3$ (double caustic).

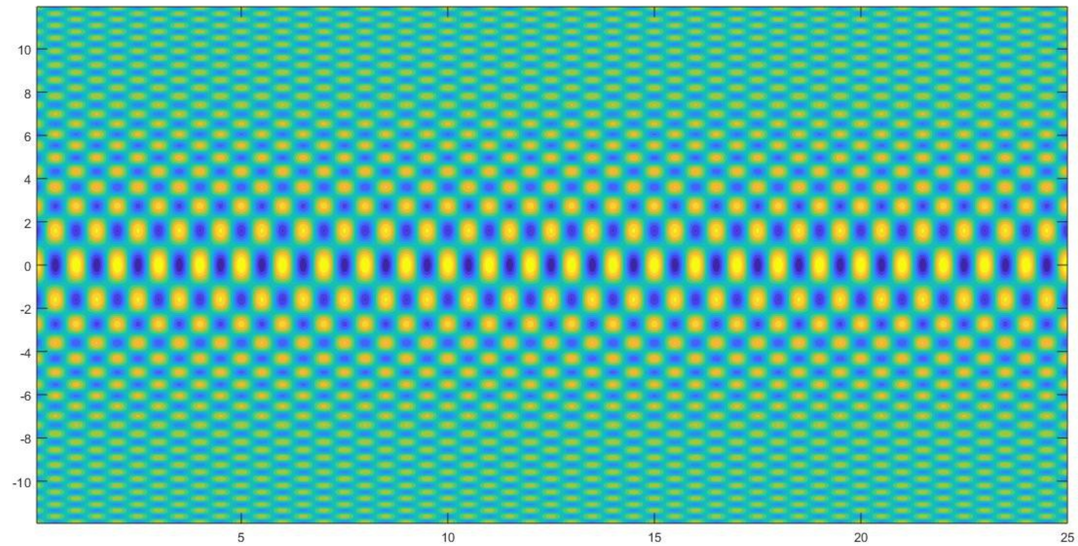


Figure 6. Scar function for $a = -3$ (transient scar).

The scar in the bowtie domain (figure 1) looks like a transient scar in our classification.

**4.** To have a better idea of scars, I drew the Wigner measure (or Husimi function) for the above 3 cases (in the transversal section in the phase space.

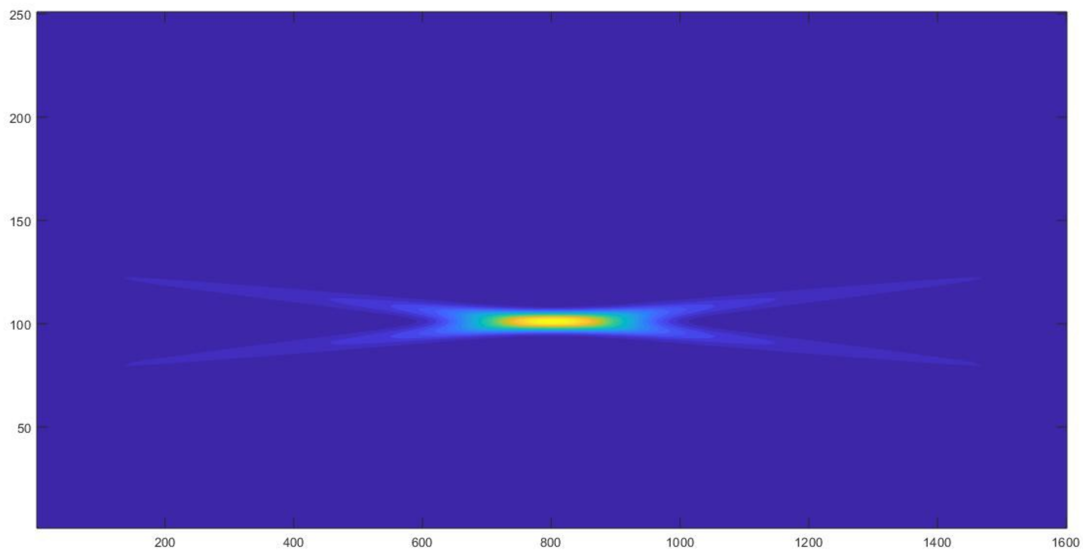


Figure 7. Wigner measure for $a = 0$ (perfect scar).

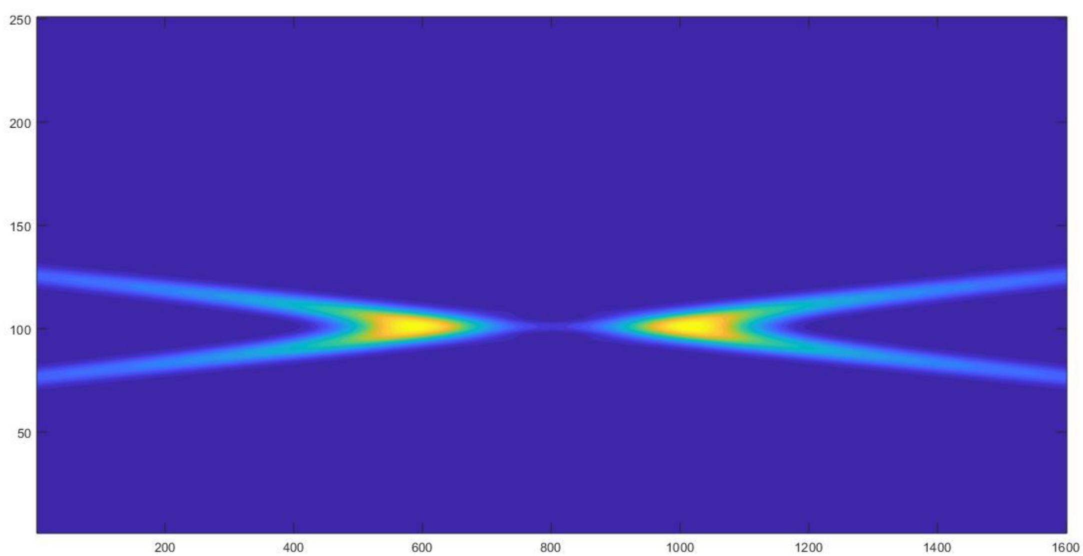


Figure 8. Wigner measure for $a = 3$ (double caustic).

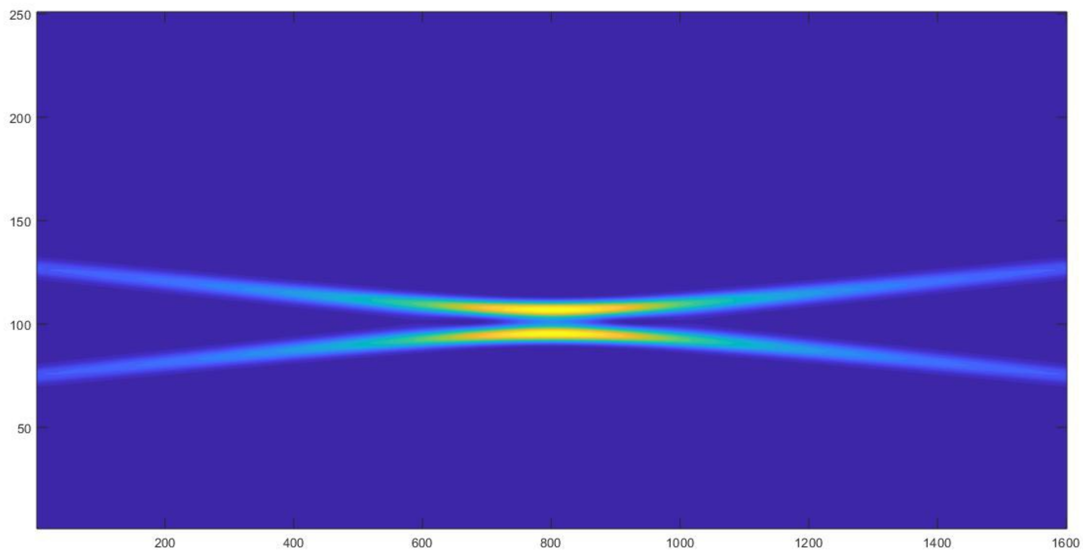


Figure 9. Wigner measure for $a=-3$ (transient scar).

The energy in the phase space is concentrated around the stable and unstable manifolds $I^s(\gamma)$ and $I^u(\gamma)$ ; it is moved by the phase flow along $I^s(\gamma)$ to $\gamma$ , and along $I^u(\gamma)$ from $\gamma$ , and on $\gamma$ the energy diffises from $I^s(\gamma)$ to $I^u(\gamma)$ :

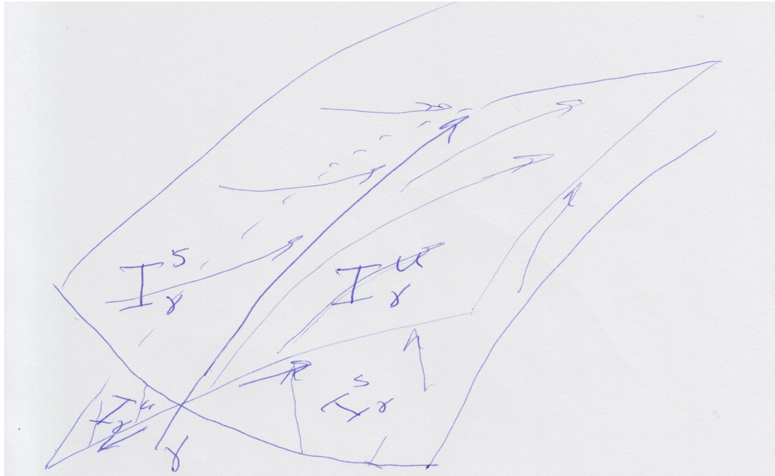


Figure 10. Stable and unstable manifolds of a trajectory $\gamma$.

So, the closed trajectory $\gamma$ works as an open resonator, or a scatterer, transforming the incoming waves (along the stable manifold $I^s(\gamma)$) into outgoing waves (along the unstable manifold $I^u(\gamma)$). It is like a sea shell, where you can hear "the sea noise":

Figure 11. Listening to the sea noise in the shell.

**5.** The above picture holds in a small neighborhood of the closed trajectory $\gamma$ which shrinks to zero as $\lambda \to -\infty$ . What farther happens to the waves, scattered by $\gamma$ ? They keep propagating along $I^u(\gamma)$ until they hit another scatterer, namely a homo- or heteroclinic trajectory. Here I have to recall some things of the hyperbolic dynamics.

Consider a compact Riemannian surface $M$ of negative curvature (to be sure, we may assume that the curvature is constant). Let $\pi_1(M)$ be its fundamental group, and $h_1, \ldots, h_{2g}$ its generators ( $g$ being the genus of $M$ ).

Consider all sequence $\hat{h}=(h_{m_i}^{\sigma_i})_{i=-\infty}^{i=\infty}$ , $\sigma_i=\pm 1$ ; let $\hat{H}$ be the set of all such sequences. For any sequence $\hat{h}\in\hat{H}$ we can define unique trajectory $\gamma(\hat{h})$ (the "shadowing property" discovered by Anosov). If the sequence $\hat{h}$ is periodic, then the corresponding trajectory $\gamma(\hat{h})$ is periodic, too.

Now suppose that $\hat{h}_1$ and $\hat{h}_2$ are two periodic sequences, and $\hat{h}=(h_{m_i}^{\sigma_i})$ is a sequence such that $\hat{h}=\hat{h}_1$ for $i<-i_0$ , and $\hat{h}=\hat{h}_2$ for $i>i_0$ for some $i_0$ (we can say that $\hat{h}$ connects $\hat{h}_1$ and $\hat{h}_2$ ). Then the trajectory $\gamma=\gamma(\hat{h})$ tends to $\gamma_1=\gamma(\hat{h}_1)$ as $t\to-\infty$ , and to $\gamma_2=\gamma(\hat{h}_2)$ as $t\to\infty$ . Such trajectory is called **heteroclinic** if $\hat{h}_1\neq\hat{h}_2$ , and **homoclinic** otherwise.

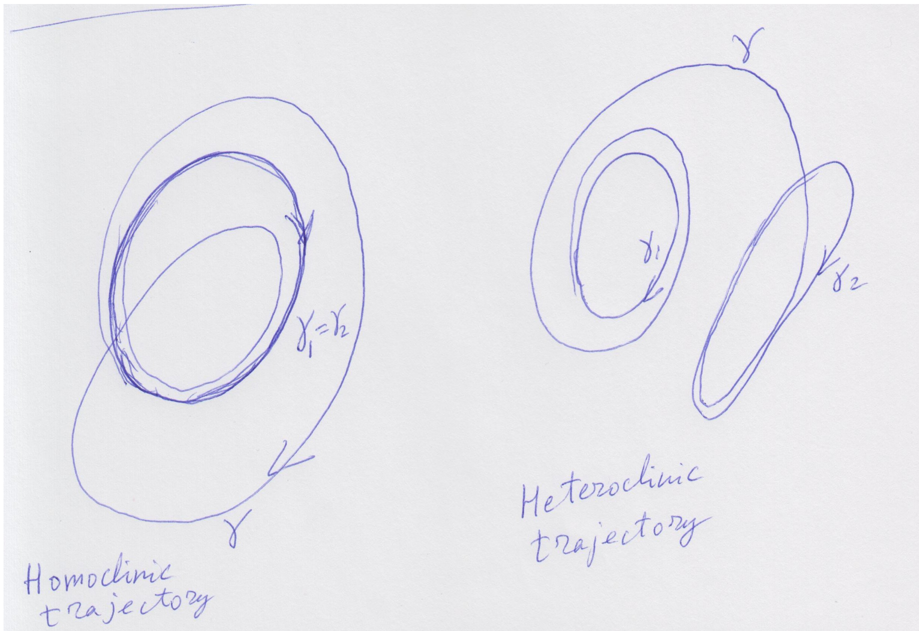


Figure 12. Homoclinic and heteroclinic trajectories.

If the sequence $\hat{h}$ connects $\hat{h}_1$ and $\hat{h}_2$ , then the trajectory $\gamma=\gamma(\hat{h})$ lies in the (transversal) intersection of $I^u(\gamma_1)$ and $I^s(\gamma_2)$ :

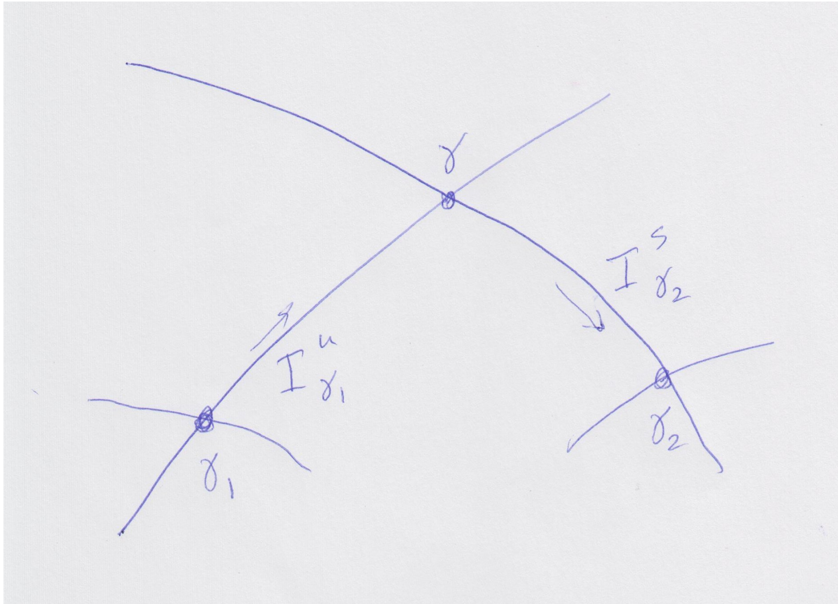


Figure 13. Heteroclinic trajectory as an intersection of unstable and stable manifolds.

The heteroclinic trajectory is an open resonator in the way similar to the closed trajectories. Hence, the waves reaching $\gamma$ along $I^u(\gamma_1)$ (which is a part of $I^s(\gamma)$ ) are scattered into $I^u(\gamma)$ which is a part of $I^s(\gamma_2)$ . Thus part of the waves scattered by $\gamma_1$ can reach $\gamma_2$ along its stable manifold.

**6.** Let $\gamma$ be a closed trajectory; let $l(\gamma)$ be its length. Then the elementary scar function near $\gamma$ corresponding to the eigenvalue $\lambda$ has the form

$$u_k^\gamma(x, y, \lambda) = \alpha_k^\gamma w_0(\eta_k, a_k) + \beta_k^\gamma w_1(\eta_k, a_k)$$

where

$w_0$ and $w_1$ are even and odd solutions of the Weber equation; $\eta_k=\sqrt{2\pi k/l(\gamma)}\, y$, and $a_k$ is such that $\lambda=-(2\pi k/l(\gamma))^2+(2\pi k/l(\gamma))a_k$. It is characterized by the vector $\begin{pmatrix}\alpha_k^\gamma\\ \beta_k^\gamma\end{pmatrix}$. In particular, if elementary scar function for the above closed trajectory $\gamma_1$ is described (for given $\lambda$) by the vector $\begin{pmatrix}\alpha_k^{\gamma_1}\\ \beta_k^{\gamma_1}\end{pmatrix}$. This wave propagates to $\gamma_2$ through the intermediate scatterer $\gamma$ and excites all elementary scar functions of $\gamma_2$: the function corresponding to the index $k'$ we have $\begin{pmatrix}\alpha_{k'}^{\gamma_2}\\ \beta_{k'}^{\gamma_2}\end{pmatrix}=A(\gamma_1,\gamma,\gamma_{2,}k,k',\lambda)\begin{pmatrix}\alpha_k^{\gamma_1}\\ \beta_k^{\gamma_1}\end{pmatrix}$ where $A(...)$ is a $2\times 2$ matrix.

Let us collect all vectors $\begin{pmatrix}\alpha_k^\gamma\\ \beta_k^\gamma\end{pmatrix}$ for all closed trajectories $\gamma$ and all $k$ in an aggregate $\zeta$; the space of these aggregates is denoted by $Z$. Let us denote by $\Xi(\gamma_1,\gamma_2)$ the set of all heteroclinic trajectories connecting $\gamma_1$ and $\gamma_2$ (homoclinic trajectories if $\gamma_1=\gamma_2$). Then we can form a matrix $S_\lambda$ whose each entry is a $2\times 2$ matrix:

$$S_\lambda(\gamma_1,k,\gamma_2,k')=\sum_{\gamma\in\Xi(\gamma_1,\gamma_2)} A(\gamma_1,\gamma,\gamma_2,k,k',\lambda)\ .$$

**7.** Now I can formulate

**Conjecture:** If $\lambda$ is an eigenvalue of Laplacian, then the equation

$$S_\lambda \zeta = \zeta$$

has a nonzero solution in the space $Z$ .

**8.** In fact, all the above pertains to a model rather than to an honest Laplacian. Instead of functions on $M$ I consider collections of scar functions around the closed trajectories. This model lies beyond the microlocal analysis, and may require new tools.

In conclusion, I'd like to note that this problem is similar to the famous Anderson transition problem. It is formulated as follows:

Consider the Schrodinger operator $\hat{H} = \varepsilon \Delta + U(x)$ , $x \in \boldsymbol{R}^3$ . Suppose $U(x)$ is a random potential, for example consisting of bumps at random locations. Then, if the concentration of bumps is lower than some threshold, the system is described by a random matrix (GUE), while if the concentration is higher above the threshold, the Anderson localization occurs. The link with our case is clear: if the concentration is low, the bumps (=scatterers) are "visible" to each other, and we can transform the problem to the model system consisting of multiple scattering problems where the analogue of our matrix $S_\lambda$ appears. In our case the "visibility" of every scatterer (closed trajectory) from any other one is absolute (it is warranted by the above symbolic dynamical description), so our case may be simpler.